TWO QUESTIONS IN ERGODIC THEORY

NISHANT CHANDGOTIA

I apologise in the beginning for this pompous piece of writing. This was written long ago (2017 perhaps) as part of an application.

1. Cocycles over Interval Exchange Transformations

This question originated from a conversation with Zemmer Kosloff who mentioned the problem in the case when $n = 4$.

Let $I := [0, 1)$ be the unit interval with the Lebesgue measure and $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a probability vector with positive entries; $n \geq 2$. Using the probability vector α we partition I into n distinct intervals $I_t; 1 \leq t \leq n$: we let

$$\begin{aligned} \beta_t &:= \sum_{s=1}^{t} \alpha_s \\ I_t &:= [\beta_{t-1}, \beta_t). \end{aligned}$$

Let π be a permutation on $\{1, 2, \ldots, n\}$; α^π is the probability vector obtained by permuting the coordinates of α via π^{-1}, that is,

$$\alpha^\pi := (\alpha_{\pi^{-1}(1)}, \alpha_{\pi^{-1}(2)}, \ldots, \alpha_{\pi^{-1}(n)}).$$

As above, we let

$$\beta_t^\pi := \sum_{s=1}^{t} \alpha_s^\pi.$$

An *interval exchange transformation* for a probability vector α and permutation π is a probability preserving transformation $T_{(\pi,\alpha)} : I \longrightarrow I$ given by

$$T_{(\pi,\alpha)}(x) := x - \beta_{t-1} + \beta^\pi_{\pi(t)-1} \text{ for } x \in I_t.$$

Informally, this corresponds to the piece-wise isometry obtained by permuting the sequence of intervals $(I_1, I_2, \ldots, I_n)$ to $(I_{\pi(1)}, I_{\pi(2)}, \ldots, I_{\pi(n)})$.

The importance of IETs cannot be overstated in contemporary mathematics with connections to several different fields and many interesting questions, some answered and many which are yet to be resolved (for a general introduction one may consider [37]).

As an (important) example, consider a cyclical permutation $\pi_{(s)}(t) = t + s \pmod n; s \leq n - 1$ and a positive probability vector $\alpha \in \mathbb{R}^n$; the corresponding interval exchange transformation (henceforth IET) $T_{(\pi,\alpha)}$ is the rotation on the circle for the angle $\sum_{t=n-s}^{n} \alpha_t$ where addition is modulo one. In fact $T_{(\pi,\alpha)}$ is a rotation if and only if π is a cyclical permutation of the aforementioned type. Automatically, if $n = 2$ then $T_{(\pi,\alpha)}$ is isomorphic to the rotation on the circle. We will assume throughout that π is *irreducible* meaning that $\pi(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$ implies $k = n$; otherwise the corresponding IET is not minimal (there are orbits which are not dense).

Having fixed an $n \geq 2$ and an irreducible permutation π, we can parametrise the IETs by the space of probability vectors with n positive entries with the appropriate Lebesgue measure. From here on, when we state an almost everywhere property of IETs we mean it with respect to this Lebegue measure (having fixed an $n \geq 2$ and an irreducible permutation). There is a vast literature on such properties of IETs: almost every IET is minimal ([21]), almost every IET is uniquely ergodic ([25], [36] and also look at [5]), IETs are never mixing [20] and almost every IET is weakly mixing (provided they are not rotations) [3] to name a few.

However there is lesser information available (and large scale interest) about skew products over IETs. Given a measurable function (called a *cocycle*) $\phi : I \longrightarrow \mathbb{R}$ and IET $T : I \longrightarrow I$, we can consider the corresponding skew-product $T^\phi : I \times \Gamma \longrightarrow I \times \Gamma$ given by

$$T^\phi(x, r) := (T(x), r + \phi(x));$$

where Γ is closed group generated by the image of ϕ; it preserves the product of the Lebesgue measure and the Haar measure on the group Γ. These constructions are of interest for several reasons; one immediate reason being that the ergodic sums of ϕ can be analysed by iterates of the map. Indeed,

$$(T^\phi)^n(x, r) := (T^n(x), r + \phi(x) + \phi(T(x)) + \phi(T^2(x)) + \ldots + \phi(T^{n-1}(x))).$$

Question 1.1. *Let $\phi : I \longrightarrow \mathbb{R}$ be a step function with integral zero. For almost every IET, $T : I \longrightarrow I$, is the skew product T^ϕ is ergodic?*

Let us briefly survey what is known about the question. In the following the characteristic function of a set $A \subset I$ is denoted by χ_A. Let us first assume that π is a permutation such that $T_{(\pi,\alpha)}$'s are rotations. By [8], the answer is affirmative in the case $\phi = \chi_{[0,\frac{1}{2})} - \frac{1}{2}$. Answering a question of Veech [35, 34], Ishai Oren proved in [27] that the answer is affirmative for $\phi = \chi_{[0,\gamma)} - \gamma; \gamma \in (0, 1)$. In [1], it was proven to be affirmative in the case ϕ is a step-function with rational discontinuities. Using ideas from [32] we believe it is an easy exercise to resolve the question completely in the case when T is a rotation.

The critical component of these proofs which assists us in resolving the question is the so-called Denjoy-Koksma inequality (consider the introduction in [24]) as a result of which we find, for a rotation by an irrational angle δ, there exists a sequence of times $t_n \in \mathbb{N}$ such that $t_n\delta$ converges to 0 and the ergodic sums of ϕ until time t_n are uniformly bounded in n. Outside some very special cases, we do not know the existence of any such inequality for IETs. The following works come to mind: by Conze and Frączek [9], Ralston and Troubetzkoy [30], Frączek and Ulcigrai [15], Hooper and Weiss [19].

2. Measures of Maximal Entropy for Hom-shifts

This question has taken inspiration from several discussions with and results by Ron Peled and Yinon Spinka and has been spelled out among us at several occasions.

In this section we will assume some basic knowledge in dynamical systems, particularly of thermodynamic formalism; with reasonable faith this prerequisite can be ignored [31, 22].

Let (X, T) be a topological dynamical system, meaning, X is a compact metric space and T is a $\mathbb{Z}^d$-action on X by means of homeomorphisms; we write $T^{\vec{i}}$ to represent the action of $\vec{i}$ on X. The study of dynamical systems (or structures in mathematics in general) is often dominated by the study of its invariants (under various notions of isomorphism). One such invariant is the topological entropy which we denote by $h_{top}(X)$. In the same vein, the measure theoretic entropy is an invariant for probability preserving transformations (X, μ, T) (meaning $T^{\vec{i}}$ preserves the measure μ) and is denoted by h_μ. Given a dynamical system (X, T), we denote the set of invariant probability measures on (X, T) by $\mathcal{M}(X)$. The *variational principle* states that

$$\sup_\mu h_\mu = h_{top}(X).$$

Under technical assumptions (which shall always hold in this section); the supremum is achieved. The measures achieving the maximum are called *measures of maximal entropy*. The dynamical systems that we care about in this section have positive entropy while IETs (as in the previous section) have zero entropy. Thus all invariant probability measures for the IETs are measures of maximal entropy.

If one *observes* a dynamical system (X, T) via the values of a function $f : X \longrightarrow \mathbb{R}$, one would like to ensure that the average long term behaviour is stable under the choice of the point; this is the content of the *ergodic theorem* given an ergodic probability measure on the space X. In addition, one would also like to avoid getting caught up in the extraneous parts of the dynamical system and hence one might wish to pick points in accordance with one of the measures of maximal entropy. It is thus to be expected that the dynamical systems (models) that we work with and consider 'natural' have at most finitely many ergodic measures of maximal entropy.

Many dynamical systems exhibit such properties and we won't be able to do justice to the vast literature on the subject in this margin space. Instead we slowly trudge towards the setting of our question. Some well known classical models where such phenomena takes place are the Ising model (look at work by Aizenmann [2], Higuchi [18] and Bodineau [4]) and the Potts model (look at [10]). The well-versed reader, who might complain saying that it does not fall in the paradigm described above, could take notice that for the Ising Model and the Potts model, due the theorems of Dobruschin [11], Lanford and Ruelle [23] the set of Gibbs states (invariant under translation of the lattice) equal the set of *equilibrium states* (with a suitable potential); equilibrium states are close cousins of the measures of maximal entropy and the results stated above are of immediate interest to us.

Let $\mathcal{H}$ be a finite undirected graph. By $\mathbb{Z}^d$, we denote both the group and its Cayley graph with respect to the standard generators. A graph homomorphism from a graph $\mathcal{G}$ to a graph $\mathcal{H}$ is a map from the vertex set of $\mathcal{G}$ to the vertex set of $\mathcal{H}$ which preserves adjacency. Let us denote the set of graph homomorphisms by $Hom(\mathcal{G}, \mathcal{H})$. The set $Hom(\mathbb{Z}^d, \mathcal{H})$ automatically forms a $\mathbb{Z}^d$- dynamical system where the set of homomorphisms is given the product topology making it a compact space and $\mathbb{Z}^d$ acts upon it by translations called the *shifts*:

$$\sigma^{\vec{i}}(x)(\vec{j}) := x(\vec{i} + \vec{j}) \text{ for all } x \in Hom(\mathbb{Z}^d, \mathcal{H}), \vec{i}, \vec{j} \in \mathbb{Z}^d.$$

These are called *hom-shifts*; they form a special yet large (and important) class of the so-called shifts of finite type [7]. Here are a few examples.

(1) The *hard-core model* is a hom-shift for the graph with vertices labelled 0 and 1 and edges $(0, 0), (0, 1)$; it consists of configurations (elements of $\{0, 1\}^{\mathbb{Z}^d}$) in the symbols $0, 1$ for which adjacent symbols can't both be 1.

(2) The *k-coloured chessboard* is a hom-shift for the complete graph with vertices labelled $1, 2, \ldots, k$; it consists of configurations (elements of $\{1, 2, \ldots, k\}^{\mathbb{Z}^d}$) in the symbols $1, 2, \ldots, k$ for which adjacent symbols are distinct.

Question 2.1. *Is it true that all hom-shifts have finitely many ergodic measures of maximal entropy?*

There are several indications towards an affirmation. We state a few. By $\mathbb{Z}^\infty$, let us denote the direct sum of a countable copies of the integers $\mathbb{Z}$. The group $\mathbb{Z}^\infty$ is amenable and $Hom(\mathbb{Z}^\infty, \mathcal{H})$ can be thought of as a 'limit' of finite-dimensional dynamical systems $Hom(\mathbb{Z}^d, \mathcal{H})$. It has been proven in [26], $Hom(\mathbb{Z}^\infty, \mathcal{H})$ have finitely many ergodic measures of maximal entropy (under the technical assumption that they are invariant under permutations of the coordinates as well).

These results have been further bolstered by recent (unpublished) work by Ron Peled and Yinon Spinka. Here they are extending on techniques (by the authors and Ohad Feldheim [29, 13, 14]) and developing many novel ideas to prove for a large class of graphs and sufficiently high dimension d, $Hom(\mathbb{Z}^d, \mathcal{H})$ have finitely many measures of maximal entropy; the class of graphs includes those corresponding to the k-coloured chessboard and the hard-core model.

One sign of encouragement is that the results of [26] and Ron Peled and Yinon Spinka seem to indicate a similar phenomena. Given a graph $\mathcal{H}$, a *phase* in $\mathcal{H}$ is an unordered pair of subsets of the graphs (A, B) such that each vertex in A is adjacent to each vertex in B. A maximal phase in $\mathcal{H}$ is one which maximises the product $|A||B|$. For instance, the maximal phase for the graph for the hard core model is $(\{0\}, \{0, 1\})$ while the AM-GM inequality implies that the maximal phases of complete graph with k vertices are precisely the disjoint sets (A, B) where $\{|A|, |B|\} = \{\lfloor\frac{k}{2}\rfloor, \lceil\frac{k}{2}\rceil\}$. Meyerovitch and Pavlov show that the measures of maximal entropy (which are invariant under the permutation of coordinates) are the ones where the partite classes of $\mathbb{Z}^\infty$ are mapped (with the uniform Bernoulli distribution) to A and B respectively. For finite (but large enough dimension) similar phenomena has been obtained by Ron Peled and Yinon Spinka under certain assumptions on the graph $\mathcal{H}$. While we do not expect the phenomena to persist verbatim to lower dimensions (in fact it does not), it still gives an indication of what one must look for.

For smaller dimensions results are fewer and sparser. For example one may look at the hard-core model for $d = 2$ ([28, Theorem 3.13] and [33]) , k-coloured chessboard for $k \geq 3.6d$ [16] ($> 4d$ follows from Dobruschin uniqueness condition as in [12], [17, Chapter 8]; probably much better bounds are known), the iceberg model [6].

Most of these results are not easy to prove; the hope is that just obtaining an upper bound on the number of ergodic measures of maximal entropy for hom-shifts (in terms of the number of maximal phases of the corresponding graph) should be more accessible.

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School of Mathematical Sciences, Tel Aviv University, Israel
E-mail address: `nishant.chandgotia@gmail.com`