

A SHORT NOTE ON THE PIVOT PROPERTY

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ABSTRACT. By \mathbb{Z}^d we mean the Cayley graph of \mathbb{Z}^d with respect to standard generators. Let $Hom(\mathbb{Z}^d, \mathcal{H})$ denote the space of graph homomorphisms from \mathbb{Z}^d to an undirected graph \mathcal{H} . We say that $Hom(\mathbb{Z}^d, \mathcal{H})$ has the generalised pivot property if there exists a finite subset $F \subset \mathbb{Z}^d$ such that for all x, y in $Hom(\mathbb{Z}^d, \mathcal{H})$ there exists a sequence of homomorphisms $x^1 = x, x^2, \dots, x^n = y \in Hom(\mathbb{Z}^d, \mathcal{H})$ for which x^i, x^{i+1} differ at most on some translate of F . In this note, we give a short introduction to the generalised pivot property followed by an example by Tim Austin of a graph \mathcal{H} for which $Hom(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property. We end with a discussion on the generalised pivot property for other subshifts like the Sturmians based on a discussion with Ville Salo.

0.1. Hom-shifts and the Generalised Pivot Property. Let \mathcal{H} denote a finite undirected graph and \mathbb{Z}^d denote both the group and its Cayley graph with respect to standard generators; by a *site* we mean an element of \mathbb{Z}^d . We denote adjacency in a graph \mathcal{G} by $\sim_{\mathcal{G}}$, that is, we say that $v \sim_{\mathcal{G}} w$ if (v, w) form an edge in \mathcal{G} . A *graph homomorphism* $x : \mathcal{G} \rightarrow \mathcal{H}$ is a vertex map such that if $i \sim_{\mathcal{G}} j$ then $x_i \sim_{\mathcal{H}} x_j$; we write x_i to mean $x(i)$. $Hom(\mathcal{G}, \mathcal{H})$ denotes the set of graph homomorphisms from \mathcal{G} to \mathcal{H} .

The set of all graph homomorphisms from \mathbb{Z}^d to \mathcal{H} , $Hom(\mathbb{Z}^d, \mathcal{H})$ is called a *hom-shift* and naturally forms a dynamical system under the shift maps: for all $\vec{i} \in \mathbb{Z}^d$, $\sigma^{\vec{i}} : Hom(\mathbb{Z}^d, \mathcal{H}) \rightarrow Hom(\mathbb{Z}^d, \mathcal{H})$ is the map

$$(\sigma^{\vec{i}}(x))_{\vec{j}} := x_{\vec{i}+\vec{j}}.$$

Thus \mathbb{Z}^d acts by homeomorphisms on $Hom(\mathbb{Z}^d, \mathcal{H})$, where the topology on the space of graph homomorphisms is the product over the discrete topology on \mathcal{H} .

This is a natural class of \mathbb{Z}^d models covering a large class of examples:

- (1) If \mathcal{H} is the graph in Figure 1 then $Hom(\mathbb{Z}^d, \mathcal{H})$ is the space of 0, 1 configurations with no two adjacent 1s. This is called the *hard-core model*.
- (2) Let K_n denote the complete graph on n vertices $\{1, 2, \dots, n\}$. Then $Hom(\mathbb{Z}^d, K_n)$ is the set of *proper n -colourings* of \mathbb{Z}^d , that is, maps from \mathbb{Z}^d to the colour set $\{1, 2, \dots, n\}$ such that adjacent colours are distinct.

In the field of symbolic dynamics, these are the so-called *vertex shifts*. Look at [4] and references within for further information on how these tie in with the field of symbolic dynamics.

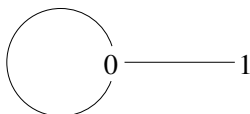


FIGURE 1. The graph for the hard-core model.

Given a graph \mathcal{H} , a *pivot move* is a pair of homomorphisms $x, y \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ which differ at most on a single site. The space of homomorphisms $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has the *pivot property* if for all $x, y \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ which differ at finitely many sites there exists a sequence of pivot moves starting from x and ending at y . We wish to address the following question:

Question: When do hom-shifts have the pivot property?

We will need the following property to be able to state a few examples: Let $N_{\mathcal{H}}(v) := \{w \in \mathcal{H} : w \sim_{\mathcal{H}} v\}$. We say that v *folds* into w if $N_{\mathcal{H}}(v) \subset N_{\mathcal{H}}(w)$ and concurrently we say \mathcal{H} folds into $\mathcal{H} \setminus \{v\}$. A graph \mathcal{H} is *bipartite-dismantlable* if there exists a sequence of successive folds $\mathcal{H} = \mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ such that \mathcal{H}_n is a single vertex, a single vertex with a self-loop or a graph with exactly two vertices joined by an edge. Any finite tree is bipartite-dismantlable because the leaves of a tree can be folded successively to obtain a smaller tree; iterating the process leads to a graph with exactly two vertices joined by an edge. Further if there is a vertex \star which is connected to all the vertices of \mathcal{H} (including itself) then it is dismantlable as well: the vertices of \mathcal{H} can be folded into \star one by one; \star is called a *safe symbol* for the space $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$. Observe that 0 is safe symbol for $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ when \mathcal{H} is the graph in Figure 1. On the other hand, none of the vertices of K_n can be folded for any n .

Folding and dismantlability were introduced in [13] to characterise cop-win graphs. It follows from the proof of [3, Theorem 4.1] that if \mathcal{H} folds into the graph \mathcal{H}' then $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has the pivot property if and only if $\text{Hom}(\mathbb{Z}^d, \mathcal{H}')$ has the pivot property as well. Before we delve into examples, we will like to remark that if \mathcal{H} is a disconnected graph with components $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_n$ then $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has the pivot property if and only if $\text{Hom}(\mathbb{Z}^d, \mathcal{H}_i)$ has the pivot property for all i . Let us look at a few examples and the main ideas involved in proving that they have the pivot property.

- (1) If \mathcal{H} is bipartite-dismantlable then $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has the pivot property; this follows easily from [3, Theorem 4.1]. Let us observe this in the simple case when $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has a safe symbol \star : If $x, y \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ differ exactly on a finite set F , we can replace symbols in $x|_F$ and $y|_F$ by \star one at a time to get a common homomorphism z . Since $\star \sim_{\mathcal{H}} v$ for all $v \in \mathcal{H}$, each intermediate configuration is also a graph homomorphism.
- (2) Let C_n denote the cycle of length n with vertices $\{0, 1, 2, \dots, n-1\}$. We say that a graph \mathcal{H} is *four-cycle hom-free* if for all homomorphisms $f : C_4 \rightarrow \mathcal{H}$ either $f(0) = f(2)$ or $f(1) = f(3)$. If \mathcal{H} does not have a self-loop then this condition is equivalent to not having C_4 as a subgraph. The graph C_3 and the graph given in Figure 1 are examples of four-cycle hom-free graphs. C_4 is a subgraph of K_4 ; it is not a four-cycle hom-free graph. It was well-known that $\text{Hom}(\mathbb{Z}^d, C_3)$ (which is the same the space of proper 3-colourings) has the pivot property. We further extended the result in [7] to prove that $\text{Hom}(\mathbb{Z}^d, C_n)$ has the pivot property for all n . In [4], it was proved that if \mathcal{H} is a four-cycle hom-free graph without self-loop then $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has the pivot property; this proof easily extends to all four-cycle hom-free graphs. One of the main idea for these proofs uses *height functions*: the so-called universal cover of a graph \mathcal{H} is a tree $U_{\mathcal{H}}$ with a canonical covering map $\pi : U_{\mathcal{H}} \rightarrow \mathcal{H}$. For instance, the universal cover of C_3 is \mathbb{Z} and the covering map is the map $\text{mod } 3$. If \mathcal{H} is a four-cycle hom-free graph then for any $x \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ there exists a unique *height function*, that is, $x' \in \text{Hom}(\mathbb{Z}^d, U_{\mathcal{H}})$ such that $\pi \circ x' = x$ up to a choice of x'_0 .

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

FIGURE 2. To show that $\text{Hom}(\mathbb{Z}^2, K_5)$ does not have the pivot property: the symbols in the rectangle can be switched but no individual symbol can be changed on its own.

- (3) The space $\text{Hom}(\mathbb{Z}^d, K_n)$ for $n \geq 2d + 2$ has the pivot property. This is well-known; for a proof look in [7, Proposition 3.4]. The following is a key idea which is used to prove this: For all $\vec{i} \in \mathbb{Z}^d$ and $x \in \text{Hom}(\mathbb{Z}^d, K_n)$ we can change x at \vec{i} to a vertex in $K_n \setminus (x(N_{\mathcal{H}}(\vec{i}) \cup \vec{i}))$ to get another homomorphism. This allows us, for instance, to remove all appearances of any chosen colour by changing one site at a time on a finite set $F \subset \mathbb{Z}^d$.

The keen reader will observe that the proof of the pivot property for three class of examples mentioned above follow from very distinct techniques. We wonder if there is some way to introduce a general technique of proof which works for the examples mentioned above.

Not all $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ has the pivot property. Consider a homomorphism in $\text{Hom}(\mathbb{Z}^2, K_5)$ be obtained by tiling the plane with the pattern given in Figure 2. It is clear that the symbols in the box can be interchanged but no individual symbol can be changed. Therefore $\text{Hom}(\mathbb{Z}^2, K_5)$ does not have the pivot property. Similarly it can be shown that $\text{Hom}(\mathbb{Z}^2, K_4)$ does not have the pivot property either.

Given a graph \mathcal{H} and a finite set $F \subset \mathbb{Z}^d$, a generalised pivot move for the shape F is a pair of graph homomorphisms $x, y \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ such that x, y differ at most on some translate of F . A space $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has the *generalised pivot property* if there is a finite subset $F \subset \mathbb{Z}^d$ such that for all $x, y \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ there exists a sequence of generalised pivot moves starting from x and ending at y for the shape F . A space having the pivot property has the generalised pivot property for $F = \{\vec{0}\}$. The following question is of interest:

Question: When do hom-shifts have the generalised pivot property?

It can be proved (but we do not prove this here) that $\text{Hom}(\mathbb{Z}, \mathcal{H})$ has the generalised pivot property for any graph \mathcal{H} . We have already seen from the examples that $\text{Hom}(\mathbb{Z}^2, K_n)$ has the pivot property for $n \neq 4, 5$. It can be shown that they have the generalised pivot property [2] (The case for $n = 4$ is from Raimundo Briceño). Let us see why this is true for $n = 5$; we indicate later in this section why does $\text{Hom}(\mathbb{Z}^2, K_4)$ have the generalised pivot property.

Observe that given a map $a : N_{\mathbb{Z}^d}(\vec{0}) \rightarrow K_5$ there exists $b \in \text{Hom}(N_{\mathbb{Z}^d}(\vec{0}) \cup \{\vec{0}\}, \mathcal{H})$ such that $b|_{N_{\mathbb{Z}^d}(\vec{0})} = a$. This is called the *single site filling (SSF)* property.

Proposition 0.1. [2] *If $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ is SSF then it has the generalised pivot property for the shape D_1 .*

First some notation, which we will use throughout the text. For all $F \subset \mathbb{Z}^d$, we will denote *the external vertex boundary of F* by

$$\partial F := \{\vec{j} \in \mathbb{Z}^d \setminus F : \vec{j} \sim_{\mathbb{Z}^d} \vec{i} \text{ for some } \vec{i} \in F\}$$

and the l^1 -ball in \mathbb{Z}^d centered at $\vec{0}$ and of radius n by D_n . For homomorphisms $x, y \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ let

$$F(x, y) := \{\vec{i} \in \mathbb{Z}^d \mid x_{\vec{i}} \neq y_{\vec{i}}\}$$

denote the set of sites where they differ.

Proof. Let $\vec{j} \in \mathbb{Z}^d$ be given and consider $\tilde{z} \in \text{Hom}(\mathbb{Z}^d \setminus (F(x, y) \cap \partial\{\vec{j}\}), \mathcal{H})$ given by

$$\tilde{z}_{\vec{i}} := \begin{cases} x_{\vec{i}} & \text{if } \vec{i} = \vec{j} \text{ or } x_{\vec{i}} = y_{\vec{i}} \\ y_{\vec{i}} & \text{if } \vec{i} \in (D_1 + \vec{j})^c. \end{cases}$$

Since $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ is SSF there exists $z \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ such that

$$z|_{\mathbb{Z}^d \setminus (F(x, y) \cap \partial\{\vec{j}\})} = \tilde{z}.$$

Observe that y and z differ at most on $D_1 + \vec{j}$ and $F(x, z) \subsetneq F(x, y)$. By induction on $|F(x, y)|$ the proof is complete. \square

The following comes from Raimundo Briceño [1]. The space $\text{Hom}(\mathbb{Z}^2, K_4)$ is not SSF but it does satisfy a more general property: Let $B_2 := \{1, 2\} \times \{1, 2\} \subset \mathbb{Z}^2$ be the induced subgraph. Given any $a \in \text{Hom}(\partial B_2, K_4)$ there exists $b \in \text{Hom}(B_2 \cup \partial B_2, K_4)$ such that $b|_{\partial B_2} = a$; there are two cases to consider:

Case(i) In this case

$$\{a_{(0,1)}, a_{(1,0)}, a_{(2,3)}, a_{(3,2)}\} = \{a_{(0,2)}, a_{(1,3)}, a_{(2,0)}, a_{(3,1)}\} = \{1, 2, 3, 4\}.$$

Then either

$$\{a_{(0,1)}, a_{(1,0)}\} \cap \{a_{(0,2)}, a_{(1,3)}\} \neq \emptyset \text{ or } \{a_{(0,1)}, a_{(1,0)}\} \cap \{a_{(2,0)}, a_{(3,1)}\} \neq \emptyset.$$

Without loss of generality assume that the former is true. It follows that

$$\{a_{(2,3)}, a_{(3,2)}\} \cap \{a_{(2,0)}, a_{(3,1)}\} \neq \emptyset.$$

Set $b_{(2,1)} \in \{a_{(0,1)}, a_{(1,0)}\} \cap \{a_{(0,2)}, a_{(1,3)}\}$ and $b_{(1,2)} \in \{a_{(2,3)}, a_{(3,2)}\} \cap \{a_{(2,0)}, a_{(3,1)}\}$. Finally set

$$b_{(1,1)} \in \{1, 2, 3, 4\} \setminus \{a_{(0,1)}, a_{(1,0)}, b_{(1,2)}, b_{(2,1)}\}, b_{(2,2)} \in \{1, 2, 3, 4\} \setminus \{a_{(2,3)}, a_{(3,2)}, b_{(1,2)}, b_{(2,1)}\} \text{ and } b|_{\partial B_2} = a.$$

It follows from the assumptions that $b \in \text{Hom}(B_2 \cup \partial B_2, \mathcal{H})$.

Case(ii) In this case either

$$\{a_{(0,1)}, a_{(1,0)}, a_{(2,3)}, a_{(3,2)}\} \neq \{1, 2, 3, 4\} \text{ or } \{a_{(0,2)}, a_{(1,3)}, a_{(2,0)}, a_{(3,1)}\} \neq \{1, 2, 3, 4\}.$$

Without loss of generality assume that the former is true. Set

$$b_{(1,1)} = b_{(2,2)} \in \{1, 2, 3, 4\} \setminus \{a_{(0,1)}, a_{(1,0)}, a_{(2,3)}, a_{(3,2)}\}.$$

Finally set

$$b_{(1,2)} \in \{1, 2, 3, 4\} \setminus \{b_{(1,1)}, a_{(0,2)}, a_{(1,3)}\}, b_{(2,1)} \in \{1, 2, 3, 4\} \setminus \{b_{(1,1)}, a_{(3,1)}, a_{(2,0)}\} \text{ and } b|_{\partial B_2} = a.$$

It follows from the assumptions that $b \in \text{Hom}(B_2 \cup \partial B_2, \mathcal{H})$.

This can be used to prove that $Hom(\mathbb{Z}^2, K_4)$ has the generalised pivot property for the shape $B_2 \cup \partial B_2 \cup \partial(\partial B_2)$.

0.2. Markov random fields, motivation and some other related questions. For us one of the key motivations for looking at the pivot property was the study of Markov random fields and Gibbs states which we now explain; for more details and background on the problem look in [6, 7, 5]. Let $\mathcal{G} = (\mathcal{A}, \mathcal{E})$ always be a locally finite undirected graph without self loops. \mathcal{A} , also referred to as the alphabet is a finite set. Given a set $A \subset \mathcal{V}$ and a pattern $a \in \mathcal{A}^A$ and $y \in \mathcal{A}^{\mathcal{V}}$ such that $y|_A = a$, we denote by

$$[y]_A := [a]_A := \{x \in \mathcal{A}^{\mathcal{V}} : x|_A = a\}.$$

the cylinder set for a . Given a probability measure μ we denote by $supp(\mu)$ the topological support of μ ,

$$supp(\mu) := \{x \in \mathcal{A}^{\mathcal{V}} : \mu([x]_A) > 0 \text{ for all finite } A \subset \mathcal{V}\}.$$

A probability measure μ on $\mathcal{A}^{\mathcal{V}}$ is called a *Markov random field* (MRF) if for all finite $A, B \subset \mathcal{V}$ satisfying $\partial A \subset B \subset A^c$ and patterns $a \in \mathcal{A}^A, b \in \mathcal{A}^B$

$$\mu([a]_A | [b]_B) = \mu([a]_A | [b]_{\partial A}) \text{ whenever } \mu([b]_B) > 0.$$

A *finite range interaction* is a function $\phi : \{a \in \mathcal{A}^A : A \subset \mathcal{V} \text{ finite}\} \rightarrow \mathbb{R}$ such that there is an n for which $\phi([a]_A) = 0$ if diameter of A is bigger than n ; a *nearest neighbour interaction* is a finite range interaction for $n = 1$. An MRF μ is a *Gibbs state with interaction* ϕ if for all $x \in supp(\mu)$

$$\mu([x]_A | [x]_{\partial A}) := \frac{e^{\sum_{C \subset A \cup \partial A} \phi(x|_C)}}{Z_{A, x|_{\partial A}}}$$

where $Z_{A, x|_{\partial A}}$ is the normalising factor.

A fundamental problem coming from statistical physics is the following: Under what assumptions on the support are the MRFs, Gibbs for some nearest neighbour interaction. We are interested in the question: Suppose μ is a shift-invariant MRF, such that $supp(\mu) = Hom(\mathbb{Z}^d, \mathcal{H})$; when is μ Gibbs for some shift-invariant nearest neighbour interaction? A celebrated theorem which brought forth such questions to the forefront is the Hammersley-Clifford theorem [9]; it implies the conclusion for $Hom(\mathbb{Z}^d, \mathcal{H})$ which have a safe symbol. This was generalised in [5] for the case when \mathcal{H} is bipartite-dismantlable. In a different direction, it was proved in [7] that the conclusion holds for $Hom(\mathbb{Z}^d, C_n)$. For $d = 1$, no assumption is required on the graph \mathcal{H} [6].

For studying this question, the formalism of Markov and Gibbs cocycles was introduced in [7] (look in it for more details); these are parametrisations of the so-called Markov and Gibbs specifications or the collections of consistent conditional probability distributions on patterns on all the finite subsets F of \mathbb{Z}^d given the pattern on ∂F . Let us now formally define cocycles (with respect to the homoclinic relation):

Let the *homoclinic relation* on $Hom(\mathbb{Z}^d, \mathcal{H})$ be denoted

$$\Delta_{\mathcal{H}}^d := \{(x, y) : x, y \in Hom(\mathbb{Z}^d, \mathcal{H}) \text{ differ at most on finitely many sites}\}.$$

A *cocycle* (with respect to the homoclinic relation) is a function $M : \Delta_{\mathcal{H}}^d \rightarrow \mathbb{R}$ such that

$$M(x, y) = M(x, z) + M(z, y) \text{ for } (x, z), (z, y) \in \Delta_{\mathcal{H}}^d.$$

A *Markov cocycle* is a cocycle where $M(x, y)$ depends only on $x|_{F \cup \partial F}, y|_{F \cup \partial F}$ where F is the set of sites on which x and y differ; in other words, if $(z, w) \in \Delta_{\mathcal{H}}^d$ is another pair such that

$$\begin{aligned} z|_{F \cup \partial F} &= x|_{F \cup \partial F} \quad , \quad w|_{F \cup \partial F} = y|_{F \cup \partial F} \\ \text{and } z|_{F^c} &= w|_{F^c} \end{aligned}$$

then $M(x, y) = M(z, w)$. Further a Markov cocycle is called *Gibbs for a nearest neighbour interaction* ϕ if

$$M(x, y) := \sum_{C \subset \mathbb{Z}^d \text{ finite}} \phi(y|_C) - \phi(x|_C) \text{ for all } (x, y) \in \Delta_{\mathcal{H}}^d;$$

the sum is well defined because there are at most finitely many sets C for which $\phi(x|_C) \neq \phi(y|_C)$.

Markov and Gibbs cocycle appear naturally in the study of the aforementioned questions: A measure μ is a shift-invariant MRF (Gibbs state with a shift-invariant nearest neighbour interaction ϕ) supported on $Hom(\mathbb{Z}^d, \mathcal{H})$ if and only if

$$M_{\mu}(x, y) := \lim_{C \rightarrow \mathbb{Z}^d} \log \frac{\mu([y]_C)}{\mu([x]_C)} \text{ for } (x, y) \in \Delta_{\mathcal{H}}^d$$

is a shift-invariant Markov cocycle (Gibbs cocycle for ϕ); an important observation which helps in understanding this is that

$$M_{\mu}(x, y) = \log \frac{\mu([y]_C)}{\mu([x]_C)}$$

where $F \cup \partial F \subset C$, F being the set of sites on which x and y differ if and only if μ is an MRF. Look at [7, Section 3] for a proof. However it is not true that every shift-invariant Markov cocycle M arises as $M = M_{\mu}$ for a shift-invariant MRF μ fully supported on $Hom(\mathbb{Z}^d, \mathcal{H})$; an example is provided in the case where $\mathcal{H} = C_n$ for $n \neq 4$ in [7].

Let $\mathbf{M}_{\mathcal{H}}^d$ and $\mathbf{G}_{\mathcal{H}}^d$ denote the space of shift-invariant Markov and Gibbs cocycles for some shift-invariant nearest neighbour interaction on $Hom(\mathbb{Z}^d, \mathcal{H})$ respectively; they form a real vector space under pointwise addition and scalar multiplication. By definition, $\mathbf{G}_{\mathcal{H}}^d \subset \mathbf{M}_{\mathcal{H}}^d$; if $\mathbf{G}_{\mathcal{H}}^d = \mathbf{M}_{\mathcal{H}}^d$ then it follows that every shift-invariant MRF supported on $Hom(\mathbb{Z}^d, \mathcal{H})$ is a Gibbs state for some shift-invariant nearest neighbour interaction. The converse is not true; we prove in [7] that

$$\dim(\mathbf{G}_{C_n}^d) = n - 1, \dim(\mathbf{M}_{C_n}^d) = n \text{ for } n \neq 1, 2, 4$$

while every shift-invariant MRF on $Hom(\mathbb{Z}^d, C_n)$ is Gibbs for some nearest neighbour interaction. If \mathcal{H} is a bipartite-dismantlable graph, then it is proved in [5] that

$$\mathbf{G}_{\mathcal{H}}^d = \mathbf{M}_{\mathcal{H}}^d.$$

Observe that the set of shift-invariant nearest-neighbour interactions on $Hom(\mathbb{Z}^d, \mathcal{H})$ forms a finite dimensional vector space (for pointwise addition and scalar multiplication) and the map taking an interaction ϕ to the corresponding Gibbs cocycle for ϕ is linear; it follows that $\mathbf{G}_{\mathcal{H}}^d$ is a finite dimensional vector space. The natural question which arises is the following: When is $\mathbf{M}_{\mathcal{H}}^d$ finite-dimensional?

It is proved in [7] that if $Hom(\mathbb{Z}^d, \mathcal{H})$ has the generalised pivot property then $\mathbf{M}_{\mathcal{H}}^d$ is a finite dimensional space: Suppose $Hom(\mathbb{Z}^d, \mathcal{H})$ has the generalised pivot property for the shape F and $M \in \mathbf{M}_{\mathcal{H}}^d$. If $(x, y) \in \Delta_{\mathcal{H}}^d$ then by the generalised pivot property it follows that

there is a sequence $x = x^1, x^2, \dots, x^n = y \in \text{Hom}(\mathbb{Z}^d, \mathcal{H})$ such that x^i, x^{i+1} differ at most on some translate of F . By the shift-invariance of M it follows that M can be determined by the values $M(z, w)$ for z, w satisfying $z|_{F^c} = w|_{F^c}$; there are finitely many distinct values by the Markov property and the shift-invariance of the cocycle. We are liable to answer the following question:

Question: Is $\dim(\mathbf{M}_{\mathcal{H}}^d)$ computable given that $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has the generalised pivot property for a given shape F ?

Under these conditions it is not hard to see that $\dim(\mathbf{G}_{\mathcal{H}}^d)$ is computable: As mentioned before the map taking a shift-invariant nearest neighbour interaction to the corresponding shift-invariant Gibbs cocycle is surjective and linear; the projection map of $\mathbf{G}_{\mathcal{H}}^d$ to pairs (x, y) which differ at most on F is a bijective linear map onto a finite dimensional vector space; the finite dimensionality is a result of the Markov property. Finally $\dim(\mathbf{G}_{\mathcal{H}}^d)$ can be computed using the rank-nullity theorem.

This was our initial motivation for looking at the generalised pivot property; similar properties have appeared in many different places. We mention a few:

- (1) Such a property was considered in [14] for possible applications to the problem of encoding data in storage systems. This is very similar to the language connectivity introduced in Section 0.4; a key difference is that the constraints defining the patterns change with the size of the box.
- (2) Such a property was also considered in [15, 17] to deduce certain invariants of tilings (by the so-called ribbon tilings) of fixed finite region R .
- (3) Suppose we were to sample a random graph homomorphism from a box $B \subset \mathbb{Z}^d$ to \mathcal{H} with respect to the uniform distribution. One of the standard techniques is to run a Markov chain on $\text{Hom}(B, \mathcal{H})$ with the following transition (Look at [12, Chapter 3] for more background): Fix a set $F \subset B$. Given $x \in \text{Hom}(B, \mathcal{H})$, choose a translate of $F + \vec{i}$ uniformly in B and resample the homomorphism x on $F + \vec{i}$ (with respect to the uniform distribution). The Markov chain (independent of the starting homomorphism) converges to the uniform distribution if and only if the corresponding Markov chain is transitive. Transitivity of this Markov chain is a property very close to the generalised pivot property but it is not clear if either implies the other.
- (4) In [18], the following problem is considered as an instance of a reconfiguration problem: Given finite graphs \mathcal{G} (without self-loops) and \mathcal{H} and $x, y \in \text{Hom}(\mathcal{G}, \mathcal{H})$ what is the complexity (in terms of $|\mathcal{G}|$) of determining whether x can be transformed into y changing one site at a time.
- (5) A similar property also appeared in [10, Section 9] in the context of building topological models. Look in Section 0.4 for more details.

0.3. Example of a Hom-Shift which does not have the Generalised Pivot Property. In this subsection we will present a construction of a graph by Tim Austin for which $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property. From the discussions in the previous subsection, the graph must have at least one four-cycle and not be too well connected at the same time (so that $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ does not have a SSF like property.)

Let \mathcal{H}_{start} be the graph given by Figure 3. Observe that \mathcal{H}_{start} is a coordinate-wise graph product of C_4 and a path graph of length five. The five copies of C_4 along the path graph have been labelled as (T, I, M, R, P) (thumb, index, middle, ring and pink) with numerical

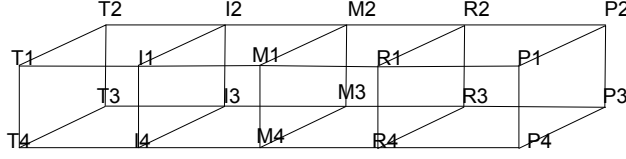


FIGURE 3. A basic component for the construction of \mathcal{H} such that $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ does not have the pivot property: \mathcal{H}_{start} .

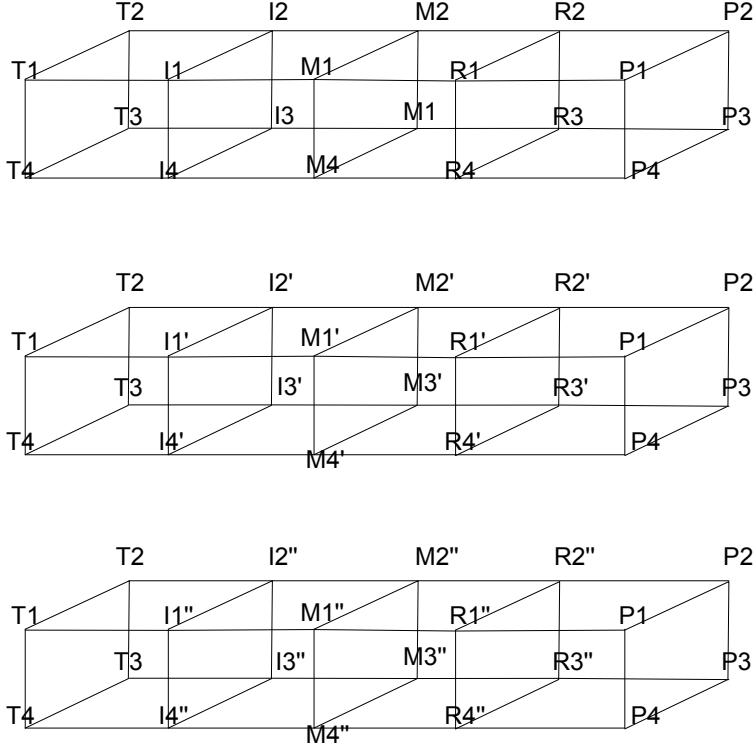


FIGURE 4. A graph \mathcal{H} for which $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property; vertices with the same label have been identified.

suffixes to indicate the position in each individual copy of C_4 . The vertices prefixed by T are the left most and the one by P are the right most copies of C_4 .

Now make two isomorphic copies of the graph \mathcal{H}_{start} calling them \mathcal{H}'_{start} and \mathcal{H}''_{start} ; for vertices labelled v in \mathcal{H}_{start} the corresponding vertices in \mathcal{H}'_{start} and \mathcal{H}''_{start} are labelled v' and v'' respectively. To obtain the graph \mathcal{H} , we make the following identifications:

- (1) The left most vertices T_i are identified with T'_i and T''_i for $1 \leq i \leq 4$.
- (2) The right most vertices P_i are identified with P'_i and P''_i for $1 \leq i \leq 4$.
- (3) The central vertices in \mathcal{H}_{start} , M_1 and M_3 are identified.

This is illustrated in Figure 4: the topmost graph is the copy of \mathcal{H}_{start} after identification; the middle and bottom are \mathcal{H}'_{start} and \mathcal{H}''_{start} respectively; vertices with the same label have been identified.

In the following we will use some special walks on the graph \mathcal{H} , for instance,

$$(M1, I1, (T1, I1', M1', R1', P1, R1'', M1'', I1'')^n, T1, I1', M1', R1', P1, R1)^\infty;$$

informally a walk like this can be thought of a walk on the 1st (the label of the vertices ends at 1) track where it starts in \mathcal{H}_{start} , cycles n times in between \mathcal{H}'_{start} and \mathcal{H}''_{start} finally returns to \mathcal{H}_{start} via \mathcal{H}'_{start} . Similar walks are considered on the 2nd, 3rd and the 4th track. We note a few important properties of the graph which we shall later utilise:

Property (1): Between any two alternate terms v, w in the walks on the graph \mathcal{H}

$$\begin{aligned} &(M1, I1, (T1, I1', M1', R1', P1, R1'', M1'', I1'')^n, T1, I1', M1', R1', P1, R1)^\infty \\ &(M2, I2, (T2, I2', M2', R2', P2, R2'', M2'', I2'')^n, T2, I2', M2', R2', P2, R2)^\infty \\ &(M1, I3, (T3, I3', M3', R3', P3, R3'', M3'', I3'')^n, T3, I3', M3', R3', P3, R3)^\infty \\ &(M4, I4, (T4, I4', M4', R4', P4, R4'', M4'', I4'')^n, T4, I4', M4', R4', P4, R4)^\infty \end{aligned}$$

there is a unique vertex adjacent to them for all n .

Property (2): Given any two vertices $v, w \neq M1 \in \mathcal{H}$ there exists at most two vertices adjacent to both of them.

Property (3): Given any vertex $v \neq M1 \in \mathcal{H}$ there exists at most three vertices adjacent to both v and $M1$.

Now fix $n \in \mathbb{N}$. We will construct distinct homomorphisms $(x^n, y^n) \in \Delta_{\mathcal{H}}^2$ such that for all $z \neq x^n \in Hom(\mathbb{Z}^2, \mathcal{H})$, the diameter of the set of sites where x^n and z differ, $diameter(F(x^n, z)) \geq 8n + 7$. The case for $n = 1$ has been illustrated in Figure 5. Set

$$\begin{aligned} &x^n|_{\{(i,0) : -4n-4 \leq i \leq 4n+3\}} \text{ as} \\ &M1, I1, (T1, I1', M1', R1', P1, R1'', M1'', I1'')^n, T1, I1', M1', R1', P1, R1 \\ &x^n|_{\{(i,1) : -4n-4 \leq i \leq 4n+3\}} \text{ as} \\ &M4, I4, (T4, I4', M4', R4', P4, R4'', M4'', I4'')^n, T4, I4', M4', R4', P4, R4 \\ &x^n|_{\{(i,2) : -4n-4 \leq i \leq 4n+3\}} \text{ as} \\ &M1, I1, (T1, I1', M1', R1', P1, R1'', M1'', I1'')^n, T1, I1', M1', R1', P1, R1 \\ &x^n|_{\{(i,3) : -4n-4 \leq i \leq 4n+3\}} \text{ as} \\ &M2, I2, (T2, I2', M2', R2', P2, R2'', M2'', I2'')^n, T2, I2', M2', R2', P2, R2 \end{aligned}$$

and then periodically tile the plane with this pattern to obtain the homomorphism $x^n \in Hom(\mathbb{Z}^2, \mathcal{H})$. For the homomorphism y^n set

$$\begin{aligned} &y^n|_{\{(i,0) : -4n-4 \leq i \leq 4n+3\}} \\ &\text{as } M1, I3, (T3, I3', M3', R3', P3, R3'', M3'', I3'')^n, T3, I3', M3', R3', P3, R3 \text{ and} \\ &y_{(i,j)}^n := x_{(i,j)}^n \text{ for } j \neq 0 \text{ or } i > 4n + 3 \text{ or } i < -4n - 4. \end{aligned}$$

We claim that for any $z \in Hom(\mathbb{Z}^2, \mathcal{H})$ either $z = x^n$ or $diam(F(x^n, z)) \geq 8n + 7$. It is suggested to the reader to play around with Figure 5 for clarity. By the periodicity of x^n we can assume without loss of generality that the northmost-eastern corner of $F(x^n, z)$ lies in the rectangle

$$\{(i, j) : -4n - 4 \leq i \leq 4n + 3, 0 \leq j \leq 3\}.$$

Set the northmost-eastern corner of $F(x^n, z)$ as (i_0, j_0) ; we will use the fact repeatedly in the following that if (i, j) is such that either $i > i_0$ or $j > j_0$ then $x_{(i,j)}^n = z_{(i,j)}$. A useful

M1	I1	T1	I1'	M1'	R1'	P1	R1''	M1''	I1''	T1	I1'	M1'	R1'	P1	R1	M1
M2	I2	T2	I2'	M2'	R2'	P2	R2''	M2''	I2''	T2	I2'	M2'	R2'	P2	R2	M2
M1	I1	T1	I1'	M1'	R1'	P1	R1''	M1''	I1''	T1	I1'	M1'	R1'	P1	R1	M1
M4	I4	T4	I4'	M4'	R4'	P4	R4''	M4''	I4''	T4	I4'	M4'	R4'	P4	R4	M4
M1	I1	T1	I1'	M1'	R1'	P1	R1''	M1''	I1''	T1	I1'	M1'	R1'	P1	R1	M1

M1	I1	T1	I1'	M1'	R1'	P1	R1''	M1''	I1''	T1	I1'	M1'	R1'	P1	R1	M1
M2	I2	T2	I2'	M2'	R2'	P2	R2''	M2''	I2''	T2	I2'	M2'	R2'	P2	R2	M2
M1	I1	T1	I1'	M1'	R1'	P1	R1''	M1''	I1''	T1	I1'	M1'	R1'	P1	R1	M1
M4	I4	T4	I4'	M4'	R4'	P4	R4''	M4''	I4''	T4	I4'	M4'	R4'	P4	R4	M4
M1	I3	T3	I3'	M3'	R3'	P3	R3''	M3''	I3''	T3	I3'	M3'	R3'	P3	R3	M1

FIGURE 5. The homomorphism x^1 is obtained by periodically tiling the \mathbb{Z}^2 -lattice with the top pattern (the pattern on the right and the upper edge are the same as ones on the left and the lower edge). The homomorphism y^1 is obtained from x^1 by changing it only on one rectangular box of dimensions 1×15 as indicated in the right pattern. The circled coordinate is the origin.

property which falls out of Property 1 is the following: Suppose $z_{(i_0, j_0)} = x_{(i_0+1, j_0+1)}^n$. Since there is a unique vertex adjacent to $x_{(i_0+1, j_0+1)}^n$ and $x_{(i_0-1, j_0+1)}^n$, it follows that $z_{(i_0-1, j_0)} = x_{(i_0, j_0+1)}^n$. By induction it follows that $z_{(i_0-k, j_0)} = x_{(i_0-k+1, j_0+1)}^n$ for all $k \in \mathbb{N}$ proving that $F(x^n, z)$ is infinite. We call this *the forcing property*.

The proof now breaks up into many cases.

Case (1): Suppose (i_0, j_0) is such that $x_{(i_0, j_0+1)}^n, x_{(i_0+1, j_0)}^n \neq M1$. Then by Property 2, it follows that $z_{(i_0, j_0)} = x_{(i_0+1, j_0+1)}^n$. By the forcing property we are done.

Case (2): Suppose (i_0, j_0) is such that $x_{(i_0, j_0+1)}^n = M1$, that is, $(i_0, j_0) = (-4n-4, 1)$ or $(-4n-4, 3)$. The two possibilities can be similarly treated; assume that $(i_0, j_0) = (-4n-4, 1)$. There are two further possibilities: Either $z_{(-4n-4, 1)} = x_{(-4n-3, 2)}^n = I1$ or $z_{(-4n-4, 1)} = I3$. The former case leads to the conclusion via the forcing property. For the latter case, we need another step. Now $z_{(-4n-5, 1)}$ is adjacent to both $z_{(-4n-4, 1)} = I3$ and $z_{(-4n-5, 2)} = R1$ and hence is equal to $M1 = x_{(-4n-4, 2)}^n$. Thus this case also leads to the conclusion via arguments very similar to the forcing property.

Case (3): Suppose (i_0, j_0) is such that $x_{(i_0+1, j_0)}^n = M1$, that is, $(i_0, j_0) = (4n + 3, 0)$ or $(4n + 3, 2)$. Again since the two possibilities can be similarly treated assume that $(i_0, j_0) = (4n + 3, 0)$. By the use of the forcing property we can assume that $z_{(4n+4-k, 0)} \neq x_{(4n+5-k, 1)}^n$ for all $k \in \mathbb{N}$. By Property 3 it follows that $z_{(4n+3, 0)} = R3$. Inductively arguing it follows that $z_{(i, 0)} = y_{(i, 0)}^n \neq x_{(i, 0)}^n$ for all $-4n - 3 \leq i \leq 4n + 3$. Thus $\text{diam}(F(x^n, z)) \geq 8n + 7$ and the proof is complete.

The following questions remain to be answered:

Question: Are there efficient ways to sample a random graph homomorphism $\text{Hom}(B, \mathcal{H})$ for a box B even when the space of graph homomorphism $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property?

Question: Is it decidable whether a hom-shift has the pivot property/generalised pivot property?

Question: Suppose $\text{Hom}(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property. Does this imply that $\text{dim}(\mathbf{M}_{\mathcal{H}}^d) = \infty$?

A related construction can be found in [7, Section 9].

0.4. Pivot property for the Sturmians. The main result (Proposition 0.2) in this section came out of discussions with Ville Salo. Another such example was constructed by Ville [16] when the question had been initially asked in [10, Section 9]. Let us begin with some basic definitions.

Fix an irrational number α and let $\mathbb{T} := [0, 1)$ in the one-dimensional torus (with addition modulo one). Consider the map $\phi : \mathbb{T} \rightarrow \{0, 1\}^{\mathbb{Z}}$ given by

$$(\phi(x))_i := (\mathbf{1}_{[1-\alpha, 1)}(x + i\alpha)) \text{ for all } i \in \mathbb{Z}$$

where $\mathbf{1}$ is the indicator function. The *Sturmian shift* for the angle α is given by

$$X_\alpha := \overline{\phi(\mathbb{T})} \subset \{0, 1\}^{\mathbb{Z}}.$$

For a shift space $X \subset \mathcal{A}^{\mathbb{Z}}$ let the *homoclinic relation* for X be given by

$$\Delta_X := \{(x, y) : x, y \in X \text{ and they differ at most on finitely many sites}\}.$$

We can now define the generalised pivot property for shift spaces X as we did for hom-shifts: X has the *generalised pivot property* if there exists n such that for all $(x, y) \in \Delta_X$ there exists a sequence $y^{(0)} = x, y^{(1)}, \dots, y^{(k)} = x \in X$ such that $y^{(i)}, y^{(i+1)}$ differ at most on some interval of length n .

Related (but it is not clear whether either implies the other) is the notion of *language connectedness* (a similar notion goes by the name Hamming connectedness [14]). Let

$$\mathcal{L}_r(X) := \{x|_{[1, r]} : x \in X\}.$$

We say that X is *language-connected* if there exists $n \in \mathbb{N}$ such that for all $r \in \mathbb{N}$ and $a, b \in \mathcal{L}_r(X)$ there exists a sequence $a = a^{(1)}, a^{(2)}, \dots, a^{(k)} = b \in \mathcal{L}_r(X)$ such that $a^{(i)}, a^{(i+1)}$ differ at most on an interval of length n .

A weaker property is to require that $a^{(i)}, a^{(i+1)}$ differ at most on n sites instead. It was asked by Mike Hochman [10, Section 9] whether there exists minimal, non-mixing shift spaces with this weaker property. The answer is an affirmative. The question arose while trying to build topologically rigid minimal model for an ergodic aperiodic rigid probability preserving system. Look at [8, 10] for more details. While the larger question still remains to be answered, we provide this simple example (minimal non-mixing subshift which is language-connected).

Proposition 0.2. *For all irrational angles α we have that the Sturmian shift X_α is language-connected.*

Remark 0.3. *One can prove quite easily in this case that the Sturmian shifts have the generalised pivot property as well. Further, the proof extends to the symbolic codings (by intervals) of minimal interval exchange transformations (look at [11] for definitions). To avoid unnecessary notational hassles we shall restrict our attention to the Sturmian shifts.*

Proof. Let $I_0 := [0, 1 - \alpha)$ and $I_1 := [1 - \alpha, 1)$. It follows from the definition of the Sturmian shifts that

$$\mathcal{L}_r(X_\alpha) = \{a \in \{0, 1\}^r : \bigcap_{1 \leq i \leq r} (I_{a_i} - i\alpha) \neq \emptyset\}.$$

The sets

$$\left\{ \bigcap_{1 \leq i \leq r} (I_{a_i} - i\alpha) : a \in \mathcal{L}_r(X_\alpha) \right\}$$

partition \mathbb{T} into intervals; enumerate it as

$$\{[c_k, c_{k+1}) : 0 \leq k \leq r\}$$

where

$$0 = c_0 < c_1 < \dots < c_{r+1} = 1.$$

Correspondingly enumerate the elements of $\mathcal{L}_r(X_\alpha)$ as $a^{(0)}, a^{(1)}, \dots, a^{(r)}$ such that

$$[c_k, c_{k+1}) = \bigcap_{1 \leq i \leq r} (I_{a_i^{(k)}} - i\alpha).$$

We will prove that $a^{(k)}, a^{(k+1)}$ differ at most on two adjacent sites.

If $a_i^{(k)} = 0, a_i^{(k+1)} = 1$ then

$$c_{k+1} = -\alpha - i\alpha.$$

If $a_i^{(k)} = 1, a_i^{(k+1)} = 0$ then

$$c_{k+1} = -i\alpha.$$

Thus we have three possible cases:

- (1) ($c_{k+1} = -i\alpha$ for some $2 \leq i \leq r$): Here $a_i^{(k)} = 1, a_i^{(k+1)} = 0, a_{i-1}^{(k)} = 0, a_{i-1}^{(k+1)} = 1$ and $a_j^{(k)} = a_j^{(k+1)}$ for $j \neq i - 1, i$.
- (2) ($c_{k+1} = -\alpha$): Here $a_1^{(k)} = 1$ and $a_1^{(k+1)} = 0$ and $a_j^{(k)} = a_j^{(k+1)}$ for $j \neq 1$.
- (3) ($c_{k+1} = -(r + 1)\alpha$): Here $a_r^{(k)} = 0$ and $a_r^{(k+1)} = 1$ and $a_j^{(k)} = a_j^{(k+1)}$ for $j \neq r$.

This completes the proof. □

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