1. Embeddings into systems with non-uniform specification

Question 1.1. [CU22] *Let $(\mathbb{T}^d, R)$ be a toral automorphism which is ergodic for the Lebesgue measure. Let (Y, S) be a homeomorphism of a Polish space without any invariant probability measure. Is there an equivariant Borel embedding of (Y, S) into $(\mathbb{T}^d, R)$?*

Hochman [Hoc19] proved that such an embedding is possible where the toral automorphism $(\mathbb{T}^d, R)$ is replaced by the full shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$ (or more generally a mixing shift of finite type). Using this, I believe it is immediate that the answer to Question 1.1 is yes when R is hyperbolic. The challenge comes from the lack of symbolic coding in the non-hyperbolic case [LS04, LS05].

Moving out of the compressible setting, in [QS16] Quas and Soo prove that for any "appropriate" Polish actions (Y, S), there exists an equivariant Borel embedding of (Y, S) into $(\mathbb{T}^d, R)$ modulo a μ-null set where μ is an invariant probability measure. To prove this they critically used the fact that ergodic toral automorphisms satisfy a nice mixing property called non-uniform specification.

Let X be a compact metric space. We say that a $\mathbb{Z}$-action, (X, T) satisfies *non-uniform specification* if there exists a sequence of increasing functions $g_n : (0,1) \to (0, \infty)$ satisfying the following conditions:

- For every $\epsilon > 0$, $\lim_{n\to\infty} g_n(\epsilon) = 0$,
- For every $n_1, \dots, n_s \in \mathbb{N}$, $i_1, \dots, i_s \in \mathbb{Z}$ and $\epsilon > 0$ such that
$$\{i_1 + (1 + g_{n_1}(\epsilon))[1, n_1], \dots, i_s + (1 + g_{n_s}(\epsilon))[1, n_s]\}$$
are pairwise disjoint, and any $x_1, \dots, x_s \in X$ there exists $x \in X$ such that $d_X(T^{i_j+t}(x), T^{i_j+t}(x_j)) < \epsilon$ for all $1 \le t \le n_j$ and $1 \le j \le s$.

Quas and Soo asked whether their result can be extended to topological dynamical systems with non-uniform specification. This was proved affirmatively by Burguet [Bur20]. Burguet proved more than what Quas and Soo were asking for: He showed that under suitable assumptions on the entropy any free Polish $\mathbb{Z}$-action (Y, S) can be embedded into a system with non-uniform specification modulo a universally null set (that is a set with measure zero for all invariant probability measures). Naturally the question arises whether embedding can be extended to the full space.

We remark that we do not even know the answer to the following question.

Question 1.2. [CU22] *Let $(\mathbb{T}^d, R)$ be a toral automorphism which is ergodic for the Lebesgue measure. Let (Y, S) be a homeomorphism of a Polish space without any invariant probability measure. Is there an equivariant Borel map of (Y, S) into the free part of $(\mathbb{T}^d, R)$?*

Along with Tom Meyerovitch we gave an alternative proof of David Burguet's result in [CM21] introducing a specification like property called flexibility. We used an approximation method to construct our embeddings, where at each stage of our construction we modify the approximate embedding on a small part of the space, and finally use Borel-Cantelli to ensure that the part of the space where our approximation doesn't converge has measure zero for any invariant probability measure. Such methods can't be used for constructing embeddings of Polish actions because there is no alternative to Borel-Cantelli lemma in this setting.

2. Conjugacy equivalence relation for ergodic $\mathbb{Z}^2$ actions

Question 2.1. *How complicated is the conjugacy relation for $\mathbb{Z}^2$ actions as compared to $\mathbb{Z}$ actions?*

Let me try to give this vague question some direction. We concentrate henceforth on the $\mathbb{Z}^2$ shift-action on the free part of the full shift $\{0,1\}^{\mathbb{Z}^2}$ which we will denote by X_2. The shift action will be denoted by σ. All that I am about to say generalises to free $\mathbb{Z}^d$ actions on Polish spaces but for simplicity we will stick to the stated setting.

Think of $\mathbb{Z}^2$ both as a group and as a directed Cayley graph with standard generators. A directed bi-infinite Hamiltonian path in $\mathbb{Z}^2$ is a connected subgraph with the same set of vertices as $\mathbb{Z}^2$ such that each vertex has exactly one incoming and one outgoing edge. There is a natural $\mathbb{Z}^2$ action on these paths by translation. Let X_H denote the space of all directed bi-infinite Hamiltonian paths.

I heard about the following result from Brandon Seward.

Theorem 2.2. [GJKS] *There is a Borel equivariant map Φ from (X_2, σ) to (X_H, σ).*

One of the consequences of this result is that it gives a (very special) orbit equivalence from (X_2, σ) to a $\mathbb{Z}$ Polish dynamical system which we denote hence forth by (X_2, T_Φ). In addition one has that if μ is an invariant ergodic probability measure for (X_2, σ) then it is also an invariant ergodic probability measure for (X_2, T_Φ). Let $\mathcal{M}_e$ denote the space of invariant probability measures for (X_2, σ). Here is a rather (wild) question.

Question 2.3. *Can Φ be chosen such that for all $\mu, \nu \in M_e$, (X_2, μ, σ) is conjugate to (X_2, ν, σ) if and only if (X_2, μ, T_Φ) is conjugate to (X_2, ν, T_Φ)?*

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