Universality in tilings: Some old results and some new

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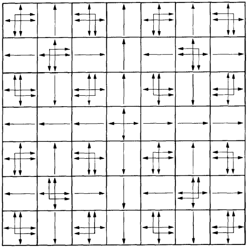
Tata institute of Fundamental Research-Centre for Applicable Mathematics

January, Algebraic and Combinatorial Invariants of Subshifts and Tilings

In this talk we will be reporting results with Tom Meyerovitch (2020), with Spencer Unger (2021) and Scott Sheffield (2021).

What kind of tilings will we look at in this talk?

There are two kinds of tilings which people study. The first kind is like Robinson's tilings.



1. They are often uniquely ergodic.
2. They are essentially minimal.
3. They have zero entropy.
4. They have no periodic points.

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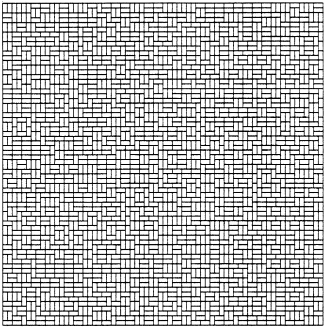
The second kind is like that of domino tilings.



1. Lot of invariant probability measures.
2. Lots of subsystems
3. Positive entropy
4. Enough periodic points to achieve the entropy.

Two kinds of tilings

Robinson's Tiling	Domino tiling
Uniquely ergodic	Lot of invariant probability measures
Essentially minimal	Lot of subsystems
Zero entropy	Positive entropy
No Periodic points	Enough periodic points to achieve the entropy.

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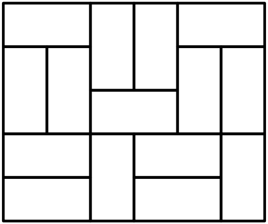
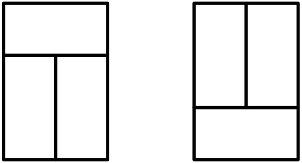


Let us observe quickly that domino tilings have a lot of probability measures on them and positive entropy.

It has a lot of invariant probability measures and positive entropy



Divide $\mathbb{Z}^d$ into a grid with rectangles of size 3×2. Consider all tilings we can obtain by arbitrarily placing one or the other tiling in the grid independently. This already tells us that the space of tilings has positive entropy and a lot of invariant probability measures.

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The entropy can be computed (Kastelyn (1961) and Temperly-Fisher(1961)). It is

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One can even compute the measure of cylinder sets for the measure of maximal entropy (Kenyon 1997) and much more (Cohn, Kenyon and Propp 2000).

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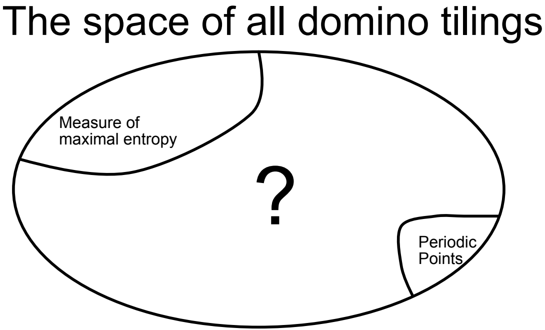

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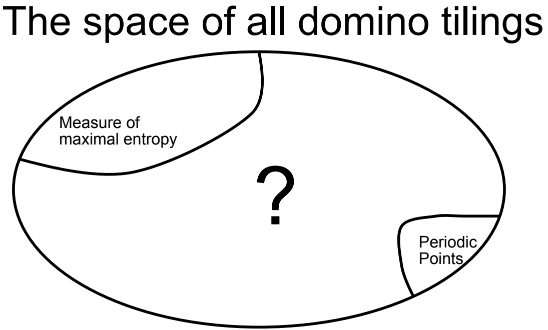

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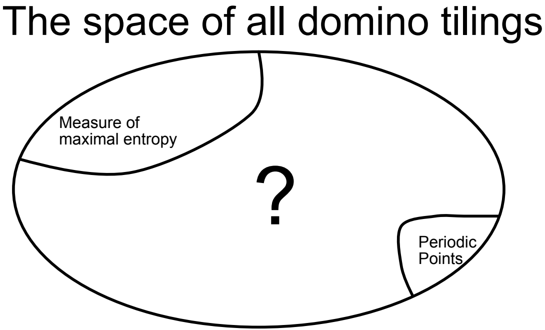

But we are left with an entire world to explore. This was one of the starting points of my work with Tom Meyerovitch.

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We denote by $h(X, T)$ the (Gurevich) entropy of (X, T), that is, the supremum of the measure theoretic entropy on X. By the variational principle, it is equal to the topological entropy when X is compact and the action is continuous.

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Suppose (X, T) is a free $\mathbb{Z}^2$ action such that

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To prove universality of a shift space we need the shift space be very flexible.

The road to flexibility

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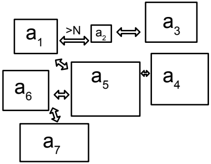

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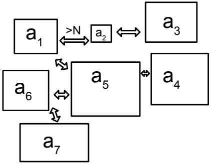

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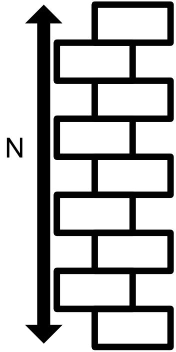

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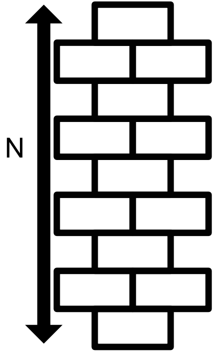

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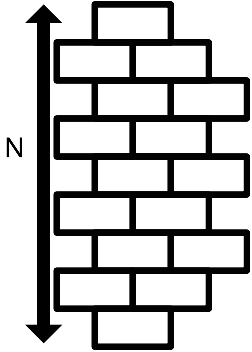

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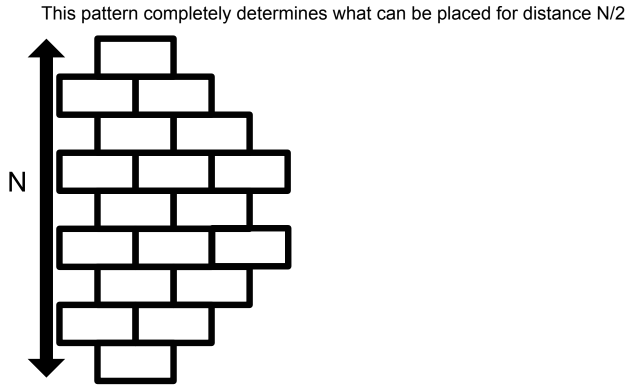

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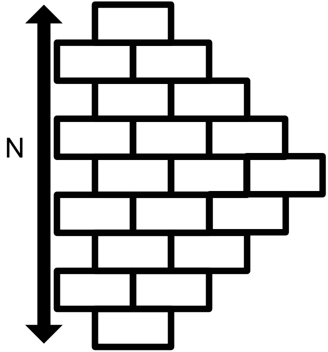

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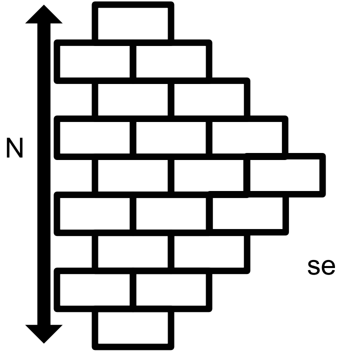

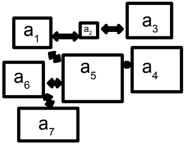

Arbitrary gluing of separated patterns can't be done

What do we need for universality: Large number of flexible patterns

It follows from work by Kastelyn (1961), Temperley-Fisher (1961) and Burton-Pemantle (1993) that the number of tilings of a $2N \times 2N$ box approximates the entropy, that is,

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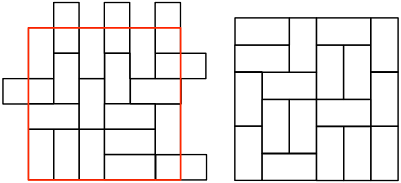

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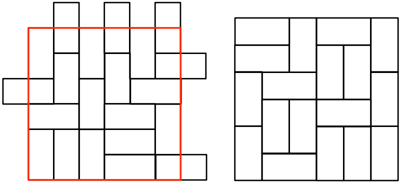

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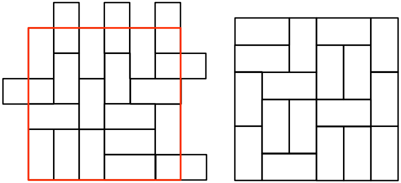
Patterns like these on boxes separated by distance 3 can be (easily) extended to a domino tiling of the $\mathbb{Z}^2$ lattice.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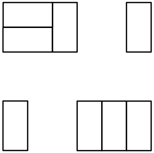

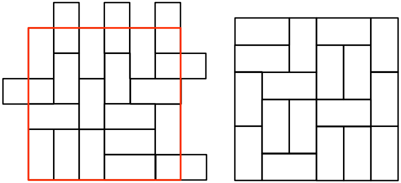

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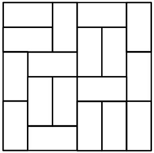

All of this can also be extended to higher dimensions.

Extension to higher dimensions

With Scott Sheffield (2021), we were able to extend this to higher dimensions. Precisely we proved that for all $d \geq 2$

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By general results from Chandgotia-Meyerovitch we have that

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Irrespective of the dimension (X^d, σ) is almost Borel universal

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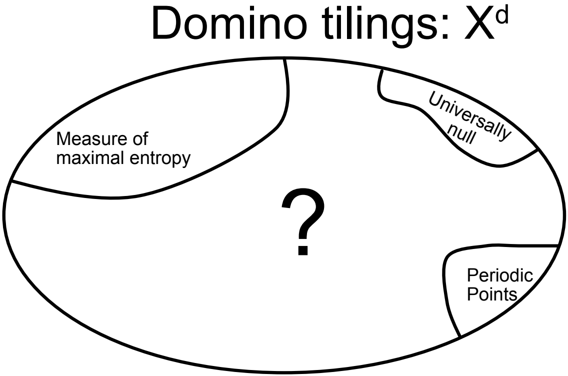
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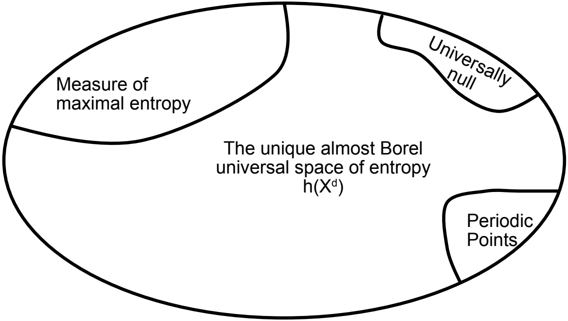

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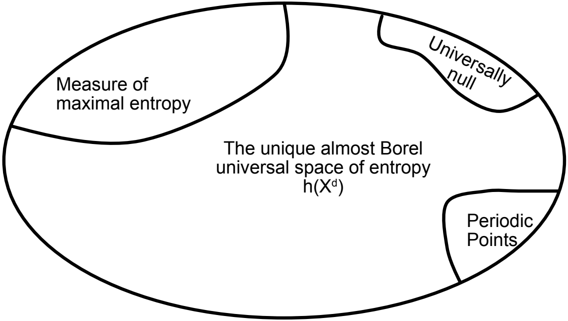

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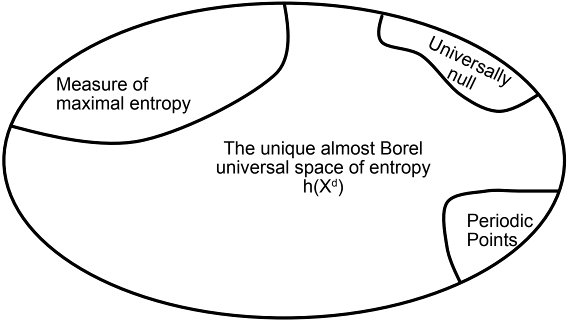

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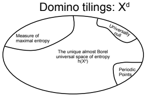
The universally null set is a very rich part of the space which carries all the infinite measures and is often very difficult to handle.

"Dark matter?"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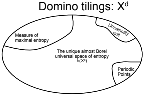

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Theorem (Hochman 2015)

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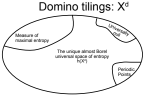

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In other words there was no need to throw away the null set. Mixing SFTs in one dimension are Borel universal.

A very curious open question

Mike Hochman mentions a wonderful open question here which is wide open.

All the maps described here are Borel.

The entropy of the full 2-shift and the proper 3-colourings of $\mathbb{Z}$ is the same.

By the previous result they are Borel isomorphic modulo the periodic points.

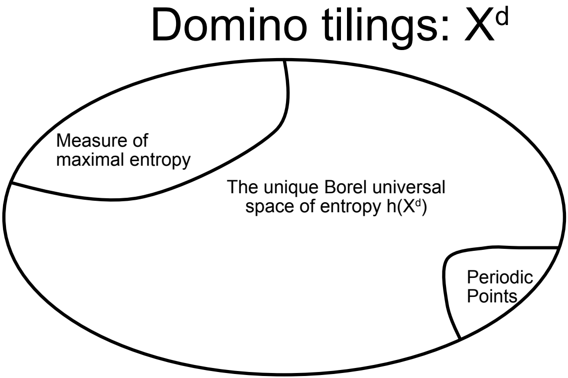
Are they topologically conjugate?

"About the dark matter?"

Theorem (Chandgotia-Unger 2021)

Suppose (X, σ) *is a* $\mathbb{Z}^d$ *shift space such that* $h(X, \sigma) < h(X^d, \sigma)$. *Then there is an equivariant embedding from free*(X, σ) *to* (X^d, σ).

Conjecture


Domino tilings: X^d
Measure of
maximal entropy
The unique Borel universal
space of entropy h(X^d)
Periodic
Points

More can be proved if we do not insist of embeddings.

Theorem (Chandgotia-Unger 2021)

Suppose (X, T) is a free $\mathbb{Z}^d$ action. Then there is an equivariant map from (X, T) to free(X^d, σ).

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The result about bi-infinite Hamiltonian paths appears in recent work by Downarowicz, Oprocha and Zhang in the ergodic category.

But how difficult does a universally null set make things?

Why is going from almost Borel universal to Borel universal so hard?

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For instance, for a free probability $\mathbb{Z}^d$ action (X, T) we have the Rokhlin's lemma which says that for all $n \in \mathbb{N}$ we can find a subset $A \subset X$ such that the tower $T^{\vec{i}}(A); |\vec{i}| < n$ almost partitions the space X (up to a small error).

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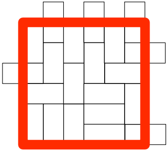
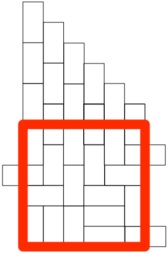
By a result of Gao, Jackson, Krohne and Seward (2015) nothing like this can hold even for very nice actions (like the free part of the full shift). They suggest a way out where the boundary of the Rokhlin towers become very "fractally"! This is an essential component of our work.

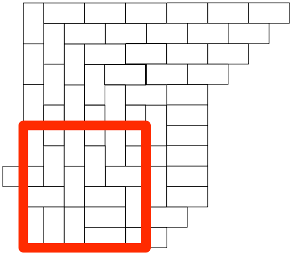
We are also missing a Shannon-McMillan theorem in this category.

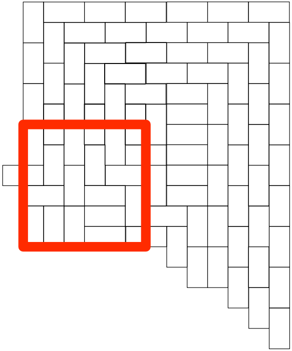
Not all the results go fully to the very general context of rectangular tilings. Let me end with some open directions.

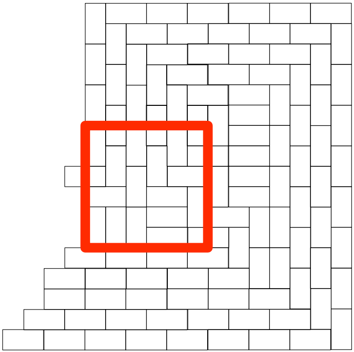
Can we always extend a tiling to that of a big box?

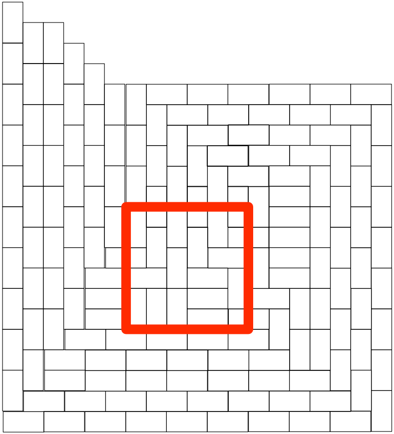


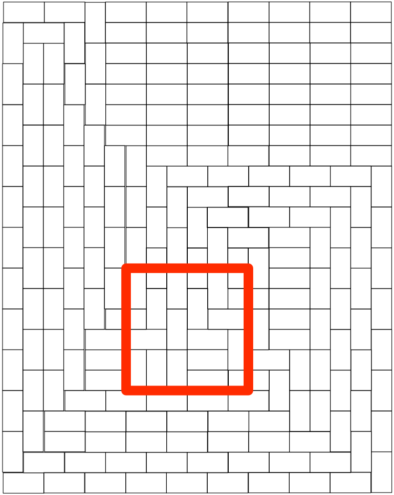












We also know this for dominos in all dimensions.

Coprime rectangular tiling shifts

Let $T_1, T_2, \ldots, T_m$ be a set of rectangles such that

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Proving that tilings by these shapes extend to tilings of a box implies topological mixing for such systems.

Coprime rectangular tiling shifts

This is known in two dimensions when there are only two tiles (Einsedler 2001). Nevertheless it should be an accessible question.

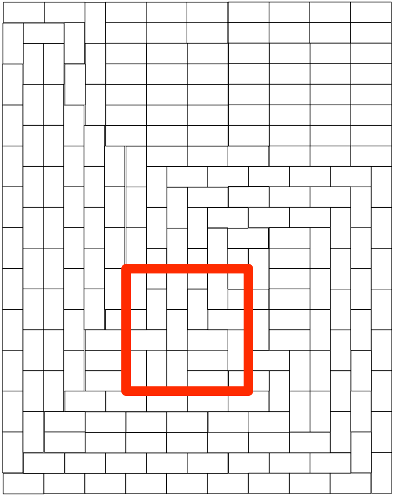
The final conjecture

Let $T_1, T_2, \ldots, T_m$ be a set of rectangles such that

gcd of the ith side length of $T_1,\ T_2,\ \ldots,\ T_m\ = 1$ for all i.

We call the space of tilings coprime rectangular tiling shifts and denote it by $X_{T_1, T_2, \ldots, T_m}$. Prove that there is a k such that

$$\frac{1}{(kN)^d} \log \left(\text{the number of tilings of a } (kN)^d \text{ box}\right) \to h_{top}(X_{T_1, T_2, \ldots, T_m}).$$



Thank you for listening.