Domino Tilings of $\mathbb{Z}^d$

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Height functions and domino tilings

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We want to study what a uniform tiling of a simply connected region by dominos looks like under various boundary conditions.

From "A Variational Principle for Domino Tilings" by Cohn, Kenyon and Propp

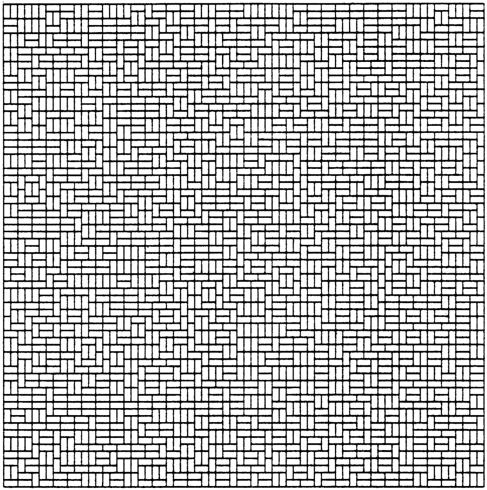

Figure : Notice that the tiling is very homogeneous in nature

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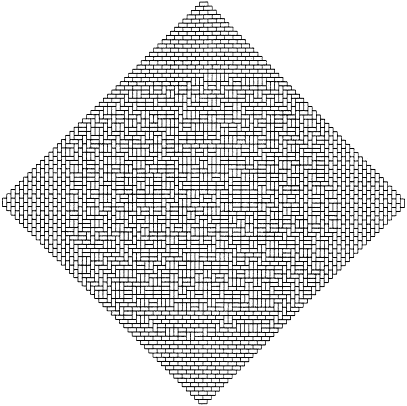


Figure : The tiling is very rigid close to the boundary and much more "random" close to the center

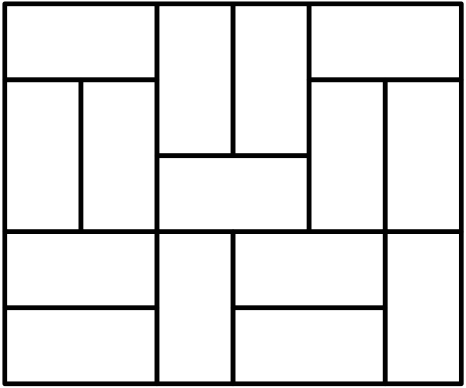
The effect of boundary conditions is, however, not entirely trivial...
(Kastelyn 1961)

$d = 2$

Let us focus on $d = 2$ where we understand a lot about this question thanks to Cohn, Kenyon and Propp.

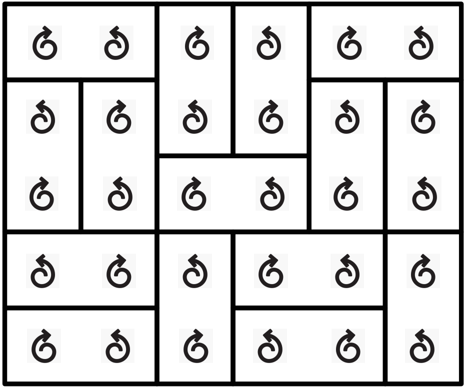
The basic stepping stone for this are height functions.

Height functions for $d = 2$



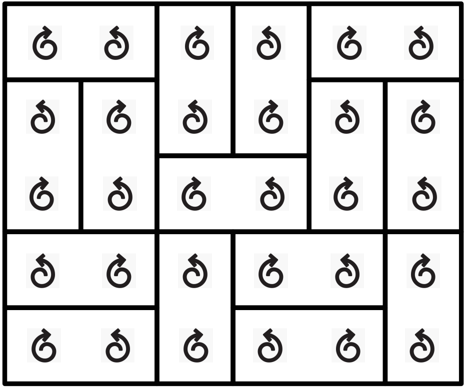
Height functions for $d = 2$

Put a clockwise spiral on even sites and an anticlocwise spiral on odd sites.



Height functions for $d = 2$

Now walk along the tiling increasing the height by 1 in the direction of the spiral.



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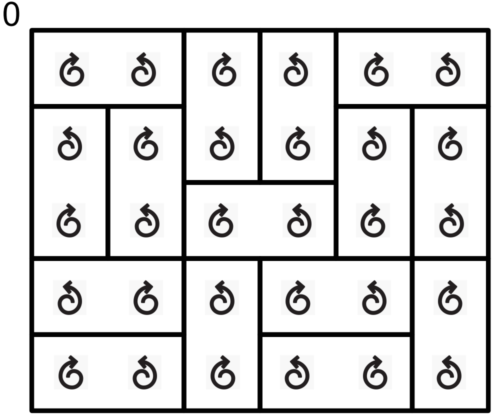

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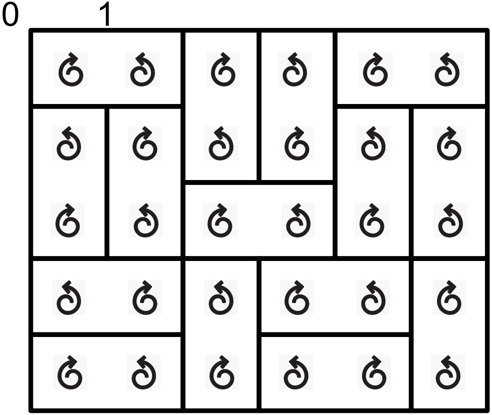

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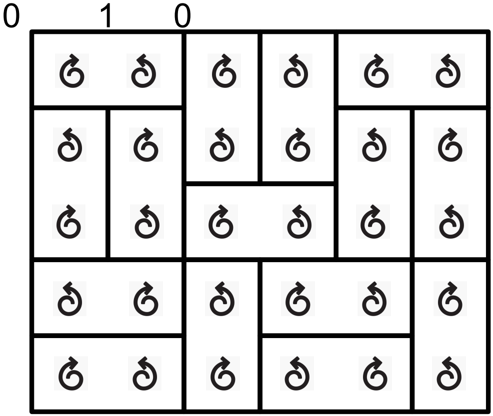

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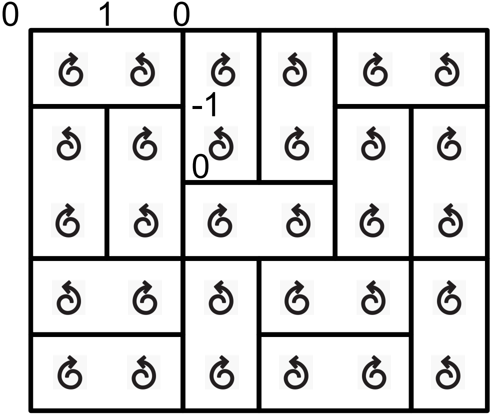

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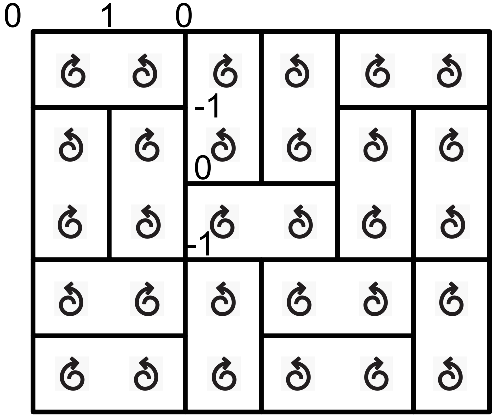

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Then the height function corresponding to a uniformly picked dimer tiling of R_n with boundary conditions h_n converges to f_{lim} in probability.

The function f_{lim} has an explicit description (which we will skip).

In particular, for nice regions R one can determine the number of dimer tilings of the approximating regions R_n with boundary conditions h_n.

This was generalised by Scott Sheffield to more general height functions (which in particular handles uniform homomorphisms to $\mathbb{Z}$).

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But first, can one define height functions for $d > 2$?

Height functions for $d = 2$

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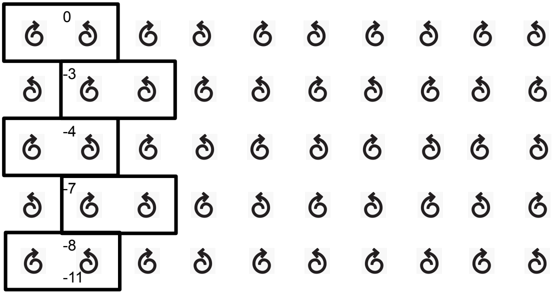

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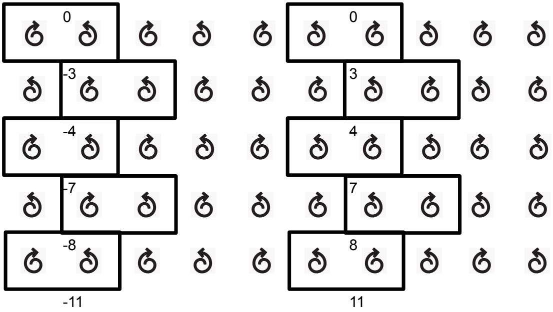

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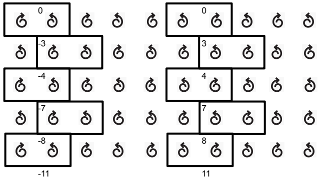

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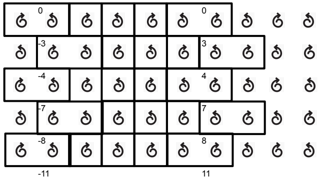
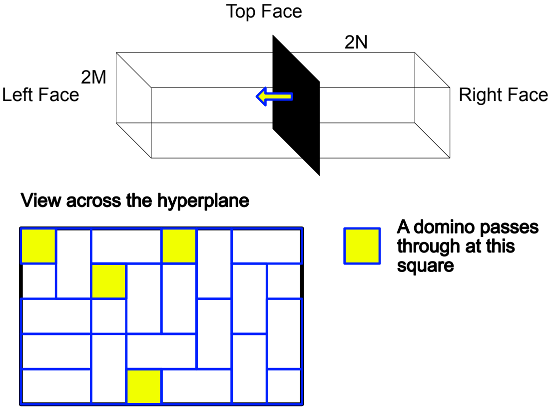


But there are substitutes.

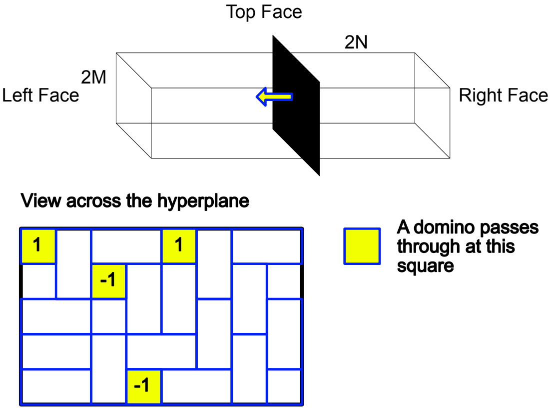
$d-1$ dimensional cocycle for tilings of $\mathbb{Z}^d$

Look at the pattern we get on the hyperplane cutting through a given tiling. Choose a normal for the hyperplane.



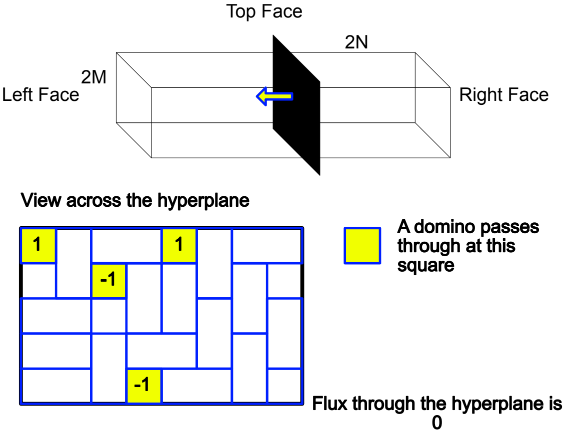
$d-1$ dimensional cocycle for tilings of $\mathbb{Z}^d$

If a domino passes from an even site to an odd site along the choice of normal, place 1 on that site. If it is against the direction of the normal, place -1 on that site.



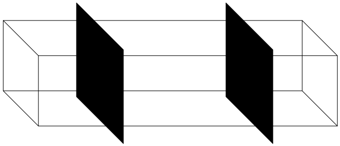
Flux

The flux passing through the hyperplane is the sum of the numbers that we get.



Flux

Flux captures how difficult it is to go from one tiling along a hyperplane to another tiling along a hyperplane.



This gives a substitute for height functions and helps us formulate appropriate conjectures for domino tilings for $d > 2$.

Conservation of Flux

For an appropriate choice of normals the net flux passing through an even sized box is zero.

Thus if the tiling is flat on all faces except the left and the right face,

the flux through the left face = the flux through the right face.

But can we prove anything with it?

A tale of two entropies

Let Box_{2n} be the tilings of $\{1, 2, \ldots, 2n\}^d$ by dominos.

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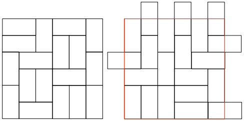

Figure : An element of Box_6 (on the left) and of All_6 (on the right)

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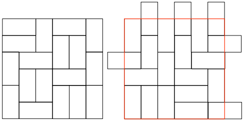

Figure : An element of Box_6 (on the left) and of All_6 (on the right)

$\mathbb{Z}^d$ acts naturally on the space of domino tilings X_{dom} by translations. The topological entropy of X_{dom} is given by the formula

$$h_{top}(X_{dom}) := \lim_{n\to\infty} \frac{1}{(n)^d} \log |All_n|.$$

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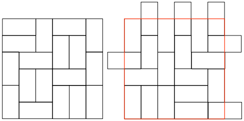

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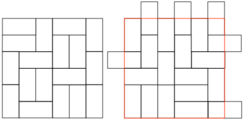
We define the box entropy as

$$h_{box}(X_{dom}) := \lim_{n\to\infty} \frac{1}{(2n)^d} \log |Box_{2n}|.$$

The main theorem



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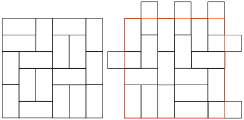

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Theorem

$h_{top}(X_{dom}) = h_{box}(X_{dom})$. Further all elements of All_n can be extended to a perfect domino tiling of a large enough box.

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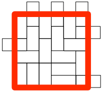

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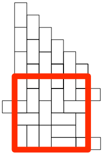

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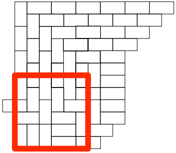

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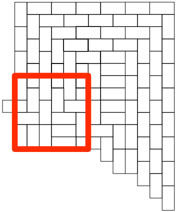

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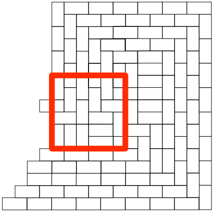

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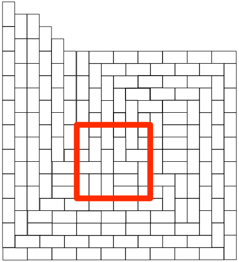

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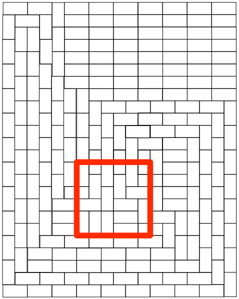


A longer introduction to the main result

Further context: Ergodic theoretic questions and their combinatorial counterparts

Our motivation comes from the ergodic theory of tiling spaces.

We will present the correspondence in a slightly more general context.

Rectangular tiling shifts

Let $\mathcal{T}$ be a set of rectangular parallelepipeds (called tiles) in $\mathbb{Z}^d$ such that for each coordinate direction, the g.c.d of the side lengths of the tiles in that direction is 1.

$X_{\mathcal{T}}$ will denote the set of tilings of $\mathbb{Z}^d$ by elements of $\mathcal{T}$.

These are called rectangular tiling shifts.

Examples:

1. Domino tilings.
2. Sets $\mathcal{T}$ which contain a singleton tile (called a monomer).

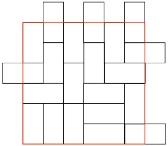
A tale of three entropies: Topological entropy

$\mathbb{Z}^d$ acts naturally on $X_{\mathcal{T}}$ by translations.

Let $All_n(\mathcal{T})$ be the patterns obtained on $\{1, 2, \ldots, n\}^d$ by restricting tilings by $\mathcal{T}$ of $\mathbb{Z}^d$.

The *topological entropy* of $X_{\mathcal{T}}$ can be calculated by the formula.

$$h_{top}(X_{\mathcal{T}}) := \lim_{n\to\infty} \frac{1}{(n)^d} \log |All_n(\mathcal{T})|.$$



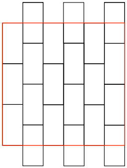
A tale of three entropies: Periodic entropy

Let N be the product of the side lengths appearing in elements of $\mathcal{T}$.

Let $Per_{nN}(\mathcal{T})$ be the set of tilings of the nN-torus $(\mathbb{Z}/nN\mathbb{Z})^d$ by elements of $\mathcal{T}$.

The *periodic entropy* of $X_{\mathcal{T}}$ is given by

$$h_{Per}(X_{\mathcal{T}}) := \lim_{n\to\infty} \frac{1}{(nN)^d} \log |Per_{nN}(\mathcal{T})|.$$



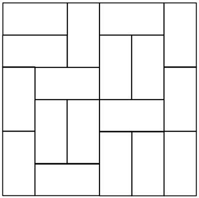
A tale of three entropies: Box entropy

Let N be the product of the side lengths appearing in elements of $\mathcal{T}$.

Let $Box_{nN}(\mathcal{T})$ be set of tilings of $\{1, 2, \ldots, nN\}^d$ by elements of $\mathcal{T}$.

The *box entropy* of $X_{\mathcal{T}}$ is given by

$$h_{Box}(X_{\mathcal{T}}) := \lim_{n \to \infty} \frac{1}{(nN)^d} \log |Box_{nN}(\mathcal{T})|.$$

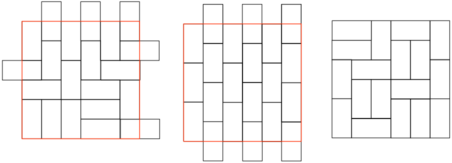


A tale of three entropies: The obvious inequalities

$$h_{top}(X_{\mathcal{T}}) := \lim_{n\to\infty} \frac{1}{(n)^d} \log |All_n(\mathcal{T})|$$

$$h_{Per}(X_{\mathcal{T}}) := \lim_{n\to\infty} \frac{1}{(nN)^d} \log |Per_{nN}(\mathcal{T})|.$$

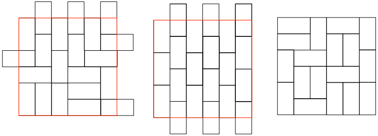
$$h_{Box}(X_{\mathcal{T}}) := \lim_{n\to\infty} \frac{1}{(nN)^d} \log |Box_{nN}(\mathcal{T})|.$$



$$All_{nN}(\mathcal{T}) \supset Per_{nN}(\mathcal{T}) \supset Box_{nN}(\mathcal{T})$$

$$h_{top}(X_{\mathcal{T}}) \geq h_{Per}(X_{\mathcal{T}}) \geq h_{Box}(X_{\mathcal{T}}).$$

A tale of three entropies: Monomers make life easy



In general

$$\begin{aligned} All_{nN}(\mathcal{T}) &\supset Per_{nN}(\mathcal{T}) \supset Box_{nN}(\mathcal{T}) \\ h_{top}(X_{\mathcal{T}}) &\geq h_{Per}(X_{\mathcal{T}}) \geq h_{Box}(X_{\mathcal{T}}). \end{aligned}$$

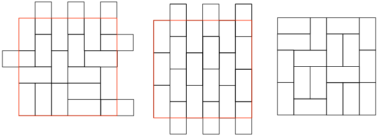
Suppose that K is the length of the longest side for a tile in $\mathcal{T}$. If $\mathcal{T}$ has a monomer then all elements of $All_{nN-K}(\mathcal{T})$ can be extended to an element of $Box_{nN}(\mathcal{T})$ and hence

$$h_{top}(X_{\mathcal{T}}) = h_{Per}(X_{\mathcal{T}}) = h_{Box}(X_{\mathcal{T}}).$$

A tale of three entropies: When is the topological entropy computable?

I conjecture that this is true for all rectangular tiling shifts $\mathcal{T}$.

Why do we care?



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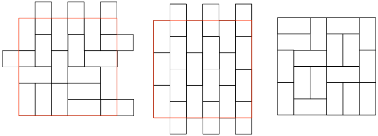

If $h_{top}(X_{\mathcal{T}}) = h_{Per}(X_{\mathcal{T}})$ then one can construct algorithms to approximate the topological entropy.

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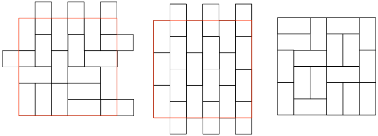

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Following my work with Tom Meyerovitch, if $h_{top}(X_{\mathcal{T}}) = h_{Box}(X_{\mathcal{T}})$ then $X_{\mathcal{T}}$ can model any probability preserving $\mathbb{Z}^d$ action (under some necessary technical constraints coming from the topological entropy).

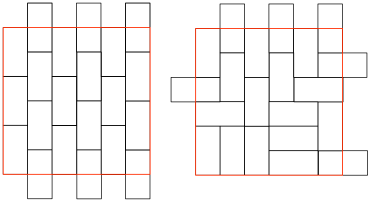
We prove this for domino tilings.

Proof of the easier result

The easier result

We will first prove the easier result: $h_{Per}(X_{dom}) = h_{top}(X_{dom})$.

For this we need to compare the size of Per_{2n} and All_{2n}.



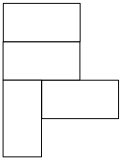
We prove that if we pick uniformly from All_{2n} then

$$\mathbb{P}(Per_{2n}) \geq \exp(-cn^{d-1})$$

for some $c > 0$.

Reflection among dominos

The main observation here is a very simple one. Domino tilings can be reflected.



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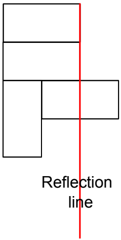

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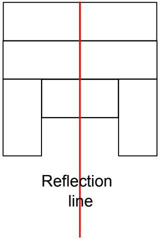

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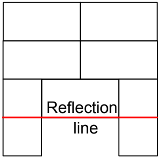

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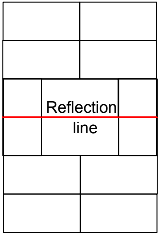

Reflection positivity

Put the uniform probability measure on All_{2N}.

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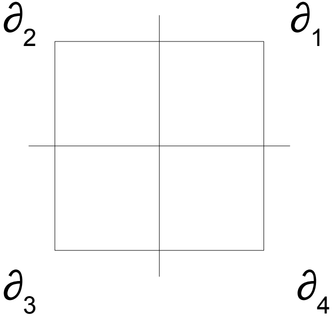
Put the uniform probability measure on All_{2N}. Divide the inner vertex boundary of $\{1, 2, 3, \ldots, 2N\}^d$ into 2^d equal parts: $\partial_1, \partial_2, \ldots, \partial_{2^d}$.

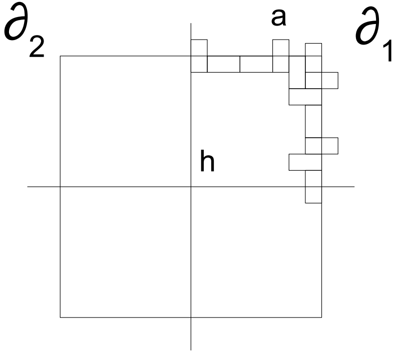


Reflection positivity: Uniform measure on All_{2N}

There exists a tiling a of ∂_1 such that

$$\mathbb{P}(a \text{ on } \partial_1) \geq e^{-cN^{d-1}}$$

for some c dependent only on d.



Reflection positivity: Uniform measure on All_{2N}; $\mathbb{P}(a \text{ on } \partial_1) \geq e^{-cN^{d-1}}$

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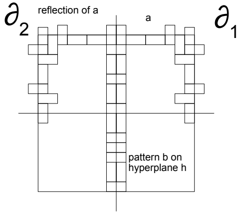
For any tiling pattern b on h we have by symmetry and by the Markov random field property,

$$\mathbb{P}(a \text{ on } \partial_1 \text{ and reflection of } a \text{ on } \partial_2 \mid b \text{ on } h) = \left(\mathbb{P}(a \text{ on } \partial_1 \mid b \text{ on } h)\right)^2 .$$



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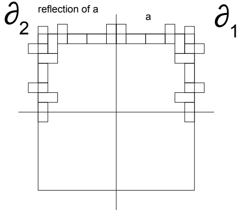
Integrating on b, we have by the Jensen's inequality



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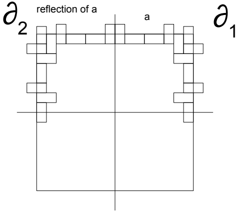
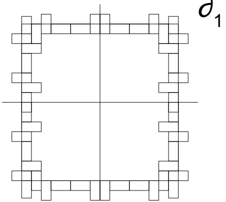


$$\mathbb{P}(a \text{ on } \partial_1 \text{ and reflection of } a \text{ on } \partial_2) \geq \left(\mathbb{P}(a \text{ on } \partial_1)\right)^2 \geq e^{-2cN^{d-1}}.$$

Reflection positivity: Periodic points have big (enough) measure

Applying reflections on $d-1$ coordinate hyperplanes



we get that

$$\mathbb{P}(Per_{2N}) \geq e^{-2^{d-1}cN^{d-1}}$$

implying that

$$\lim_{N\to\infty} \frac{1}{(2N)^d} \log|Per_{2N}| = \lim_{N\to\infty} \frac{1}{(2N)^d} \log|All_{2N}|.$$

The entropy of domino tilings is computable

Thus $h_{top}(X_{dom}) = h_{Per}(X_{dom})$. The entropy of X_{dom} is a computable number.

Reflection positivity is an old and highly specialised technique; it doesn't apply to any other tiling model.

It was used recently also by Lorenzo Taggi to show that correlation between monomers in the dimer model ($d > 2$) does not decay.

It was not important in this calculation that these be tilings of $\{1, 2, \ldots, 2N\}^d$. We could prove a similar estimate for any box all of whose side lengths are even.

The (slightly more) harder part:

$$h_{top}(X_{dom}) = h_{Box}(X_{dom}).$$

Introducing long rectangles

We need to prove that

$$\lim_{N\to\infty} \frac{1}{(2N)^d} \log |Box_{2N}| = \lim_{N\to\infty} \frac{1}{(2N)^d} \log |All_{2N}|.$$

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$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{(2N)^d} \log |Box_{2N}| &= \lim_{N,M\to\infty} \frac{1}{(2N)(2M)^{d-1}} \log |Box_{2N,2M}| \\
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Pick uniformly from $All_{2N,2M}$. It is sufficient to prove that

$$\mathbb{P}(Box_{2N,2M}) \geq \exp(-C_M(M^{d-2}N + M^{d-1})).$$

All but the left and right faces are flat

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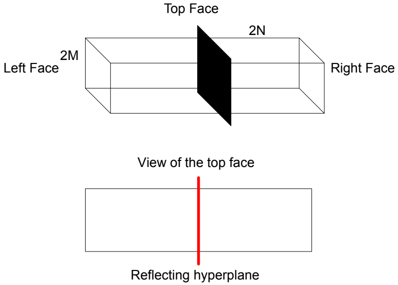
Observe that when we reflect along hyperplanes parallel to the smaller face on all other faces the tiling becomes flatter and flatter.



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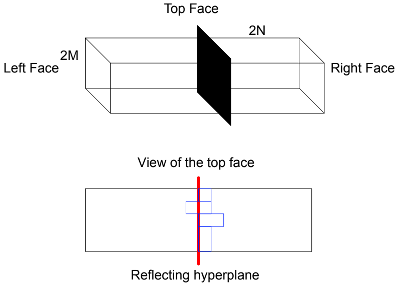

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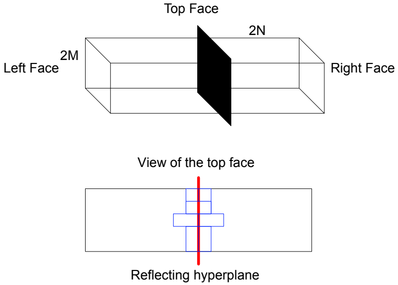

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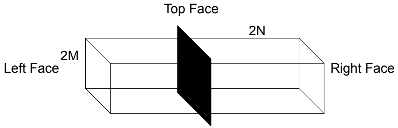

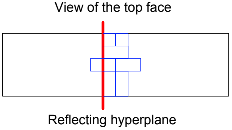

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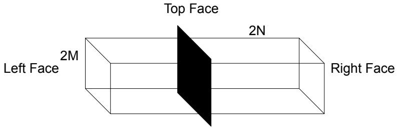

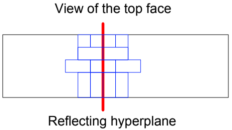

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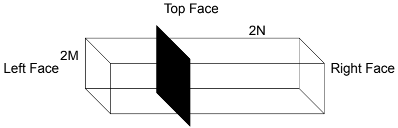

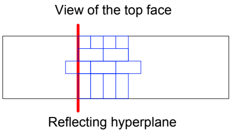

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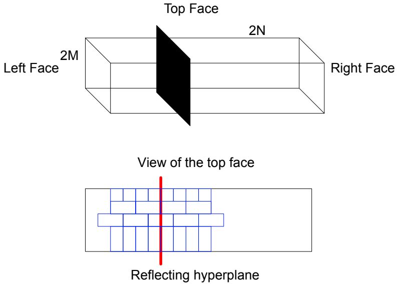

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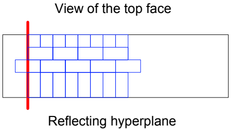

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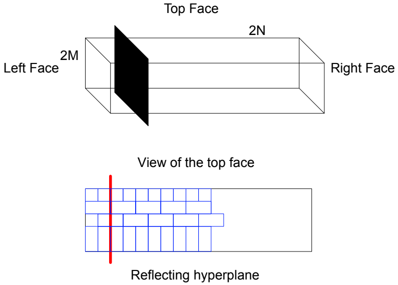

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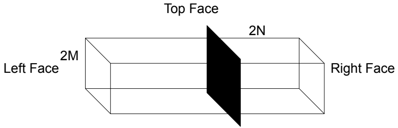

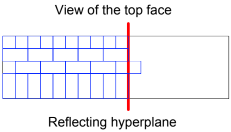

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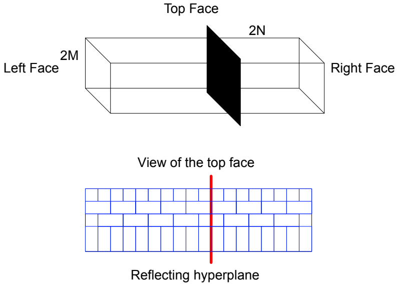

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All but the left and right faces are flat

By reflection positivity, for a uniform tiling from $All_{2N,2M}$ we have

$$\mathbb{P}(\text{all but the left and right faces are flat}) \geq \exp(-c_M(M^{d-2}N)).$$

for some $c_M > 0$.

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Recall: $d - 1$ dimensional cocycle for tilings of $\mathbb{Z}^d$

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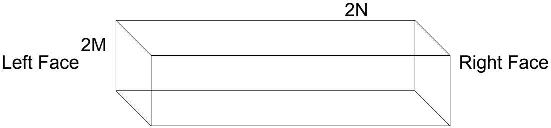
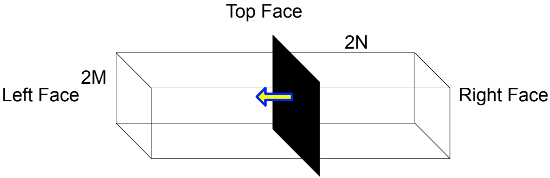
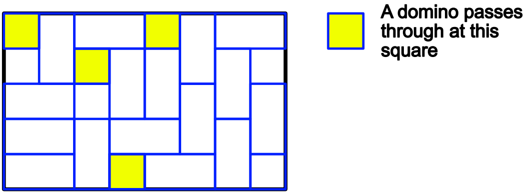


Recall: $d-1$ dimensional cocycle for tilings of $\mathbb{Z}^d$

Now look at the pattern we get on the hyperplane cutting through a given tiling. Choose a normal for the hyperplane.

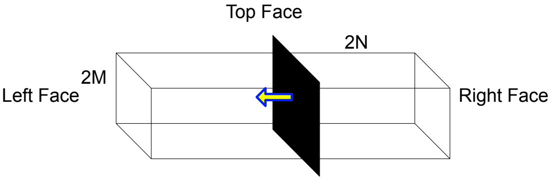


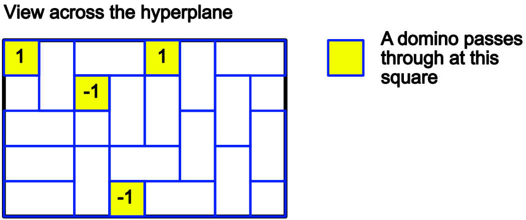
View across the hyperplane



Recall: $d-1$ dimensional cocycle for tilings of $\mathbb{Z}^d$

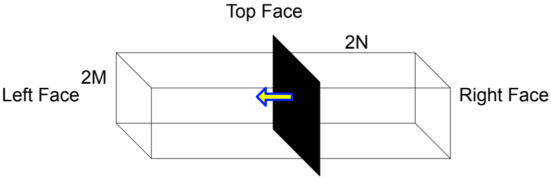
If a domino passes from an even site to an odd site along the choice of normal, place 1 on that site. If it is against the direction of the normal, place -1 on that site.

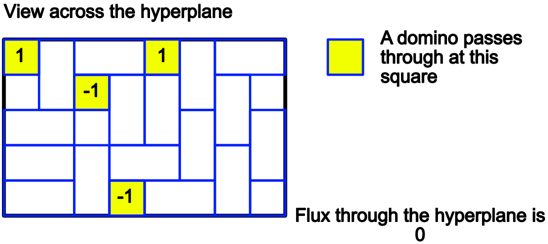




Recall: Flux

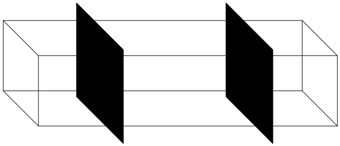
The flux passing through the hyperplane is the sum of the numbers that we get.





Recall: Flux

Flux captures how difficult it is to go from one tiling along a hyperplane to another tiling along a hyperplane.



Recall: Conservation of Flux

For an appropriate choice of normals the net flux passing through an (N, M)-rectangle is zero.

Thus if the tiling is flat on all faces except the left and the right face,

the flux through the left face $=$ the flux through the right face.

Reflection positivity is back: Zero flux

By reflecting along the sides and applying reflection positivity we can assume that the flux through the left (and hence the right face) is zero.

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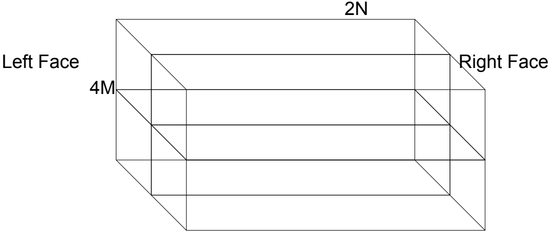


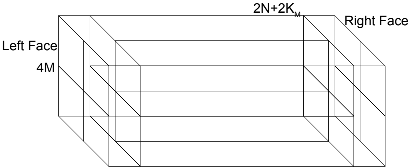
From all that we have said, if a tiling is picked uniformly from $All_{2N,4M}$ then

$$\mathbb{P}(\text{zero flux}) \geq \exp(-C_M(M^{d-2}N + M^{d-1})).$$

Irreducibility of equal flux component

Once the flux is fixed, the set of tiling patterns that we can see on the left face form an irreducible chain.

Thus we can lengthen the $(N, 2M)$-rectangle to a $(N + K_M, 2M)$ box and obtain an extension of any zero flux tiling to an element of $Box_{2N+2K_M,4M}$.

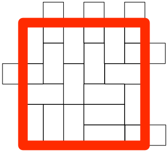


This completes the proof that

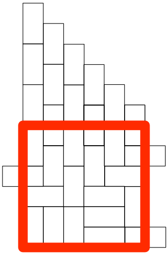
$$\mathbb{P}(Box_{2N+2K_M,4M}) \geq \exp(-C_M(M^{d-2}N + M^{d-1})).$$

Extending an element of All_{2N} to a tiling of a box

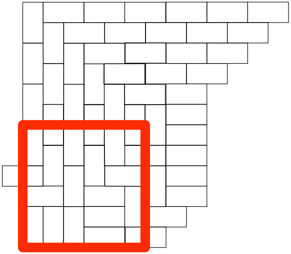
First make all but one face flat



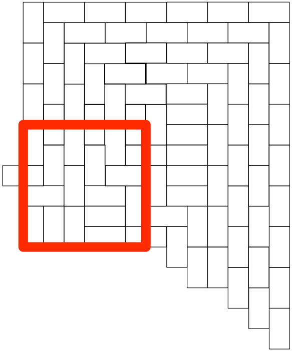
First make all but one face flat



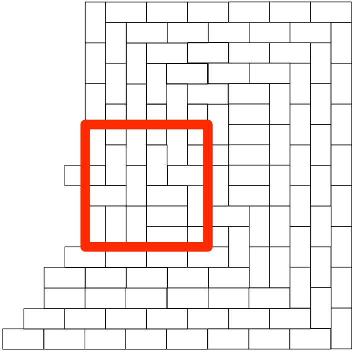
First make all but one face flat



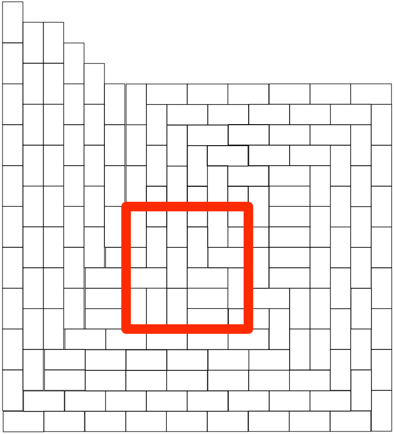
First make all but one face flat



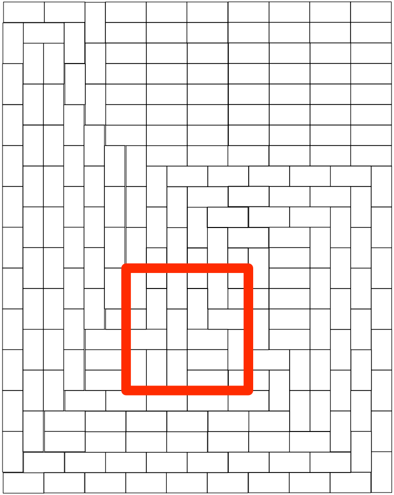
First make all but one face flat



First make all but one face flat



Finally use the irreducibility of flux components to make the last face flat



Summary

1. Domino tilings can model all free ergodic random fields of appropriate entropy.
2. The box-entropy coincides with the topological entropy for domino tilings.
3. All elements of All_{2N} can be extended to a tiling of a large enough box.
4. There is an analogue to height functions for higher dimensional domino tilings (which we call flux).

Future directions

1. Other rectangular tiling spaces!
2. Prove a variational principle.

Thank you