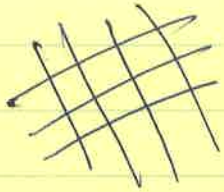
KANSAS & DENVER

jw Brian Marcus.

$\mathcal{H}$ - undirected graph.

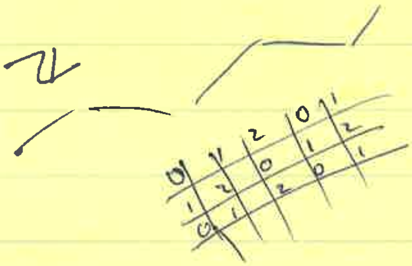
$\mathbb{Z}^2$

$\mathcal{G}$ - undirected graph without self-loops.



$$\mathrm{Hom}(\mathcal{G},\mathcal{H}) := \{x : \mathcal{G} \to \mathcal{H} \mid i \sim_{\mathcal{G}} j \Rightarrow x_i \sim_{\mathcal{H}} x_j\}$$

$\mathbb{Z}$

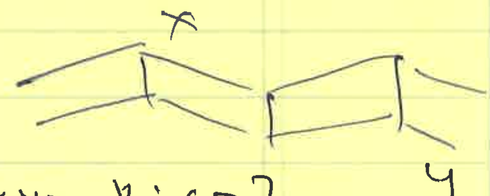


$\mathcal{H}$ - 3  - proper 3-colouring of ~~$\mathbb{Z}^d$~~ $\mathcal{G}$.

$\mathcal{H}$ - 0 — 1 - ~~Hard~~ no two 1's are adjacent. (hard core model)

$$\mathcal{H}_{walk} = (\mathrm{Hom}(\mathbb{Z},\mathcal{H}), \mathcal{E}_{walk})$$

$$\mathcal{E}_{walk} = \{(x,y) : x_i \sim y_i \ \forall i \in \mathbb{Z}\}.$$



Main Qu: When is $\mathrm{diam}(\mathcal{H}_{walk}) < \infty$?

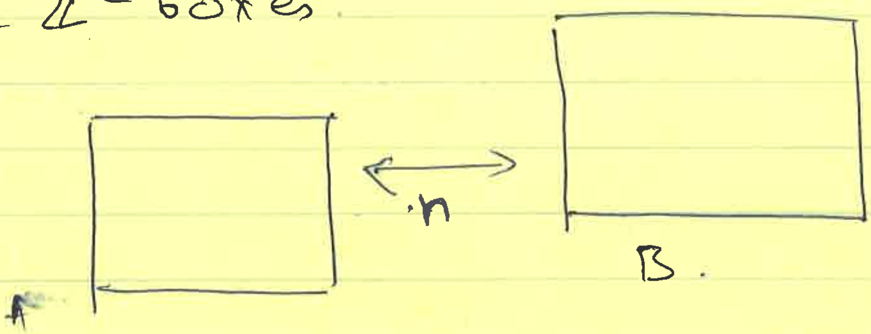
Motivation: $\mathrm{Hom}(\mathbb{Z}^2,\mathcal{H})$ forms a dynamical system. (translation of say $x \in \mathrm{Hom}(\mathbb{Z}^2,\mathcal{H})$ is still a homomorphism).

Want to address → How do properties of $\mathcal{H}$ reflect in the dynamics of $\mathrm{Hom}(\mathbb{Z}^2,\mathcal{H})$?

By properties $\rightsquigarrow$ mixing properties.

What is a mixing property?

$A, B \subset \mathbb{Z}^2$ boxes



$a \in \mathrm{Hom}(A, H)$, $\quad b \in \mathrm{Hom}(B, H)$

(Does there exists $x \in \mathrm{Hom}(\mathbb{Z}^2, H)$ s.t. $x|_A = a$, $x|_B = b$?) (*)

n depends on a, b $\longrightarrow$ transitivity

(*) $\rightarrow$ holds for all H-connected

n independent of ~~A, B~~ a, b $\rightarrow$ block-gluing.

<u>Qn:</u> When is $\mathrm{Hom}(\mathbb{Z}^2, H)$ block-gluing?

Note if H is bipartite.

a, b are ~~sin~~ patterns on A, B singletons

then n depends on whether a, b are of the same partite class.

So H-bipartite $\Rightarrow$ $\mathrm{Hom}(\mathbb{Z}^2, H)$ not block-gluing

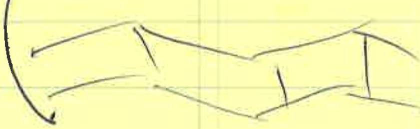
③

n independent of A, B. — phased block-gluing
(might depend on phase of a, b.)

$\mathrm{Hom}(\mathbb{Z}^d, H)$ is ~~block~~ phased block-gluing
iff
$\mathrm{diam}(H_{\mathrm{walk}}) < \infty$

$\uparrow$

$\boxed{\text{Recall}}$ (Hom $\mathbb{Z}$, H.



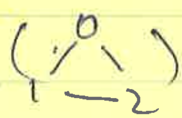
__Examples:__

$(012)^\infty$

$\leftarrow$ 0
1
2
0
1
2
0
1
2
,
,
,

Anything at finite distance ~~essentially look like this~~. to $(012)^\infty$ looks like $(012)^\infty$

— distance$((012)^\infty, (01)^\infty) = \infty$.

$\mathrm{diam}\Big(\big($  $\big)_{\mathrm{walk}}\Big) = \infty$

H

x_1	0
x_2	0
$\vdots$	$\vdots$
x_n	0

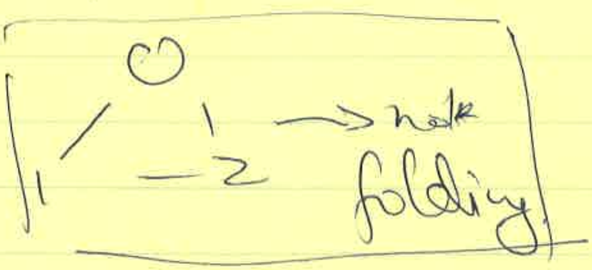
$0^\infty \sim x$ for all $x \in \mathrm{Hom}(\mathbb{Z}, H)$

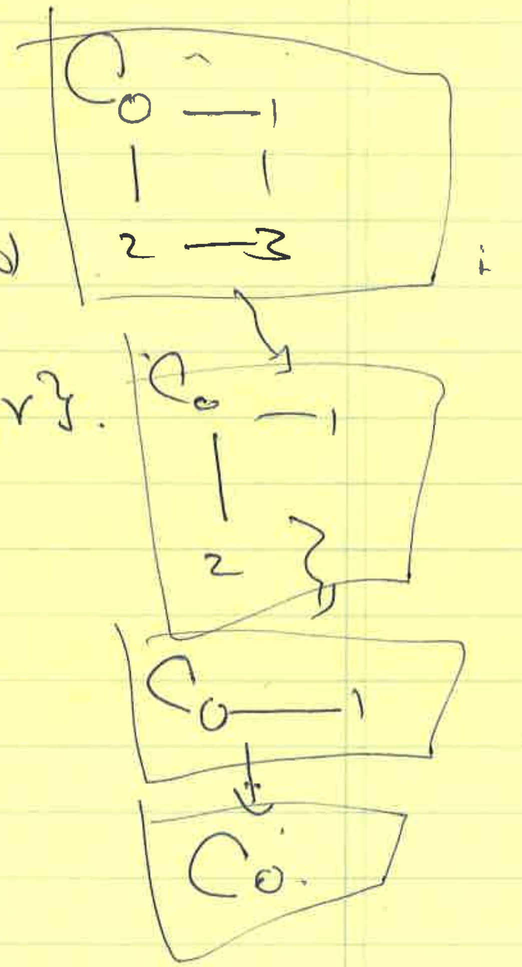
$\mathrm{diam}(0 - 1)_{\text{walk}} = 2$

More generally, H-graph.

v folds into w if $N_H(v) \subset N_H(w)$

$N_H(v) = \{u \in H : u \sim_H v\}$.





$x \in \mathrm{Hom}(\mathbb{Z}, H)$

shift of $x \sim x$

R

x_2	x_1
x_1	x_0
x_0	x_{-1}
x_{-1}	x_{-2}
x_{-2}	x_{-3}.

Can replace all v's by w to get $y \sim x$ where $y \in \mathrm{Hom}(\mathbb{Z}, H \setminus \{v\})$

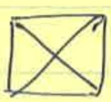
H is called bipartite dismantlable if $\exists$ sequence of folds starting with H and ending with 

Thm: H bip. dismantlable $\Rightarrow$ $\mathrm{diam}(H_{walk}) < \infty$

Thm: ~~H has~~ Let H be undirected graph without self loops and copies of 

then $\mathrm{diam}(H_{walk}) < \infty$ iff H is not ~~bip~~ bip. dismantlable.

Remark: Converse is not true in general.

$\rightarrow$ K_4  $= H$ $\quad \mathrm{diam}(H_{walk}) \leq 6$.

$\rightarrow$ Ronnie Pavlov

Conjecture: It is undecidable whether $\mathrm{diam}(H_{walk}) < \infty$.

How to prove. $\mathrm{diam}(H_{walk}) = \infty$

for $H =$ 

There is a natural map. $\bmod 3 : \mathbb{Z} \to C_3 = H$

This induces a covering map from

$(\mathbb{Z})_{walk}$ to $(H)_{walk}$.

But distance $\left(\; , \; \right) = \infty$

difference of height is n

$\to$ difference of height ≤ 1

number of steps $\geq n/2$ for all n.

$\Rightarrow$ $\mathrm{diam}((\mathbb{Z})_{walk}) = \infty$

$\Rightarrow$ $\mathrm{diam}(H_{walk}) = \infty$.

If H ~~is~~ has no self loops and four cycles use π: universal cover of $H \to H$.

Thm: It is decidable whether $\operatorname{diam}(H_{walk}) \leq n$

Conjecture: It is undecidable whether

$\operatorname{diam}(H_{walk}) < \infty$.