Predictive Sets

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December 2021, MIT

Can you guess?

Suppose I give you a sequence: 1,

Can you guess?

Suppose I give you a sequence: 1, 1,

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Can you guess?

Suppose I give you a sequence: 1, 1, 1, 1, 1, 1,

What comes next?

Can you guess?

Suppose I give you a sequence: 1, 1, 1, 1, 1, 1,

What comes next?

It is probably going to be 1.

Can you guess?

What about this one: 1,

Can you guess?

What about this one: 1, 2,

Can you guess?

What about this one: 1, 2, 3,

Can you guess?

What about this one: 1, 2, 3, 1,

Can you guess?

What about this one: 1, 2, 3, 1, 2,

Can you guess?

What about this one: 1, 2, 3, 1, 2, 3,

Can you guess?

What about this one: 1, 2, 3, 1, 2, 3,

It is probably going to be 1 again.

Can you guess?

What about this one: 1, 2, 3, 1, 2, 3,

It is probably going to be 1 again.

But it could very well have been part of

$$\ldots, 4, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 1, 2, 3, 4, \ldots$$

in which case it should have been 4.

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We know that without enough information about how the sequence comes about there is not much point in guessing.

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But what if instead I give you the entire past of the sequence and tell you before hand that the sequence is periodic. Then we can always predict precisely.

Do we need to know the entire past to make this prediction?

Predicting periodic sequences

Clearly, it would be enough to know the sequence along the even integers because the restriction of periodic sequence to the even integers is still periodic.

$$\ldots, 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots$$

$$\color{red}{?}, \times, 1, \times, 3, \times, 1, \times, 3, \times, 1, \times, 3, \times, \ldots$$

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Clearly it is not enough to know the sequence along the odd integers.

$$?, 1, \times, 3, \times, 1, \times, 3, \times, 1, \times, 3, \times, \ldots$$

We do not know after all which periodic sequences runs along the odds.

Can we cut down further?

Predicting periodic sequences

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Even integers are PERIODIC* but odd integers are not PERIODIC*.

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Suppose $x_i; i \in \mathbb{Z}$ is a periodic sequence with period p.

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Now suppose that x_i is constant for $i \in P \cap \{nk \ : \ k \in \mathbb{N}\}$ for some $n \in \mathbb{N}$. But P is PERIODIC*. Hence it also contains a multiple of np.

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Now suppose that x_i is constant for $i \in P \cap \{nk \ : \ k \in \mathbb{N}\}$ for some $n \in \mathbb{N}$. But P is PERIODIC*. Hence it also contains a multiple of np.

Hence we can decide what x_0 is, given $x_i; i \in P$.

In other words, a set can predict all periodic sequences if and only if it is PERIODIC*.

A similar statement holds for processes arising from compact group rotations in general.

Entropy and Prediction

By a process we mean a stationary process with a finite state space unless stated otherwise.

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Shannon entropy of a process is a measure of how unpredictable a process is. Indeed, the Shannon entropy,

$$h(X_{\mathbb{Z}}) := H(X_0 \mid X_{\mathbb{N}}) = 0$$

if and only if X_0 is measurable with respect to $X_{\mathbb{N}}$.

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Warning: The formula $h(X_{\mathbb{Z}}) = H(X_0 \mid X_{\mathbb{N}})$ is true only for finite valued processes. There are infinite entropy Gaussian processes which can be predicted by their past.

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A stationary Gaussian process has zero entropy if and only if its spectral measure does not have an absolutely continuous component.

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$\mathbb{N}$ is a predictive set.

$k\mathbb{N}$ is predictive

The process $X_{\mathbb{Z}}$ has zero entropy if and only if $X_{k\mathbb{Z}}$ has zero entropy.

Thus P is a predictive set if and only if kP is also a predictive set.

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On the other hand $P = k\mathbb{N} + r$ can not even predict periodic sequences for r which is not a multiple of k.

$P = k\mathbb{N} + r$ is not predictive (when r is not a multiple of k

In fact there exist zero entropy weak mixing processes (think of this as a certain decay of correlation assumption) $X_{\mathbb{Z}}$ such that X_0 is independent of $X_{k\mathbb{N}+r}$.

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Question

Suppose P is a predictive set. By definition for all zero entropy processes $X_{\mathbb{Z}}$

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Does there exists $n \in \mathbb{N}$ such that for all zero entropy processes $X_{\mathbb{Z}}$

$$H(X_{-n} \mid X_P) = 0?$$

Some sufficient conditions.

Return-time sets are predictive

Given a process $Y_{\mathbb{Z}}$ with U in its state space, we write

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$k\mathbb{N}$ is the return time for the periodic process

$$U_1, U_2, \ldots, U_k, U_1, U_2, \ldots, U_k, U_1, U_2, \ldots, U_k, \ldots$$

This generalises our observation that $k\mathbb{N}$ is a predictive set.

An example of a predictive set

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$$\{n \ : \ n\alpha \mod 1 \in (-\epsilon, \epsilon)\}.$$

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In fact if P is predictive then

$$P \cap \{n \ : \ n\alpha \mod 1 \in (-\epsilon, \epsilon)\}$$

is also predictive.

Some necessary conditions

$SIP^\star$ sets

Given a sequence of natural numbers $S = \{s_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$, we write

$$SIP(S) := \left\{ \sum_{i=1}^{\infty} \epsilon_i s_i \;:\; \epsilon_i \in \{-1, 0, 1\} \right\} \cap \mathbb{N}.$$

A set $P \subset \mathbb{N}$ is called $SIP^\star$ if it intersects every SIP set.

$SIP^\star$ sets

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1. $k\mathbb{N}$ is $SIP^\star$: Given a sequence $S = \{s_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$ there exists a subsequence $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ (which are equal modulo k) such that
$$\sum_{t=1}^{k} s_{i_t} \in k\mathbb{N}.$$
Thus $SIP(S) \cap k\mathbb{N} \neq \emptyset$.

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Thus $SIP(S) \cap k\mathbb{N} \neq \emptyset$.
2. if $r \not\equiv 0 \pmod{k}$ then $k\mathbb{N} + r$ is not SIP^*: If a sequence $S \subset k\mathbb{N}$ then $SIP(S) \subset k\mathbb{N}$ and $SIP(S) \cap (k\mathbb{N} + r) = \emptyset$.

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3. $SIP^\star$ sets have bounded gaps.

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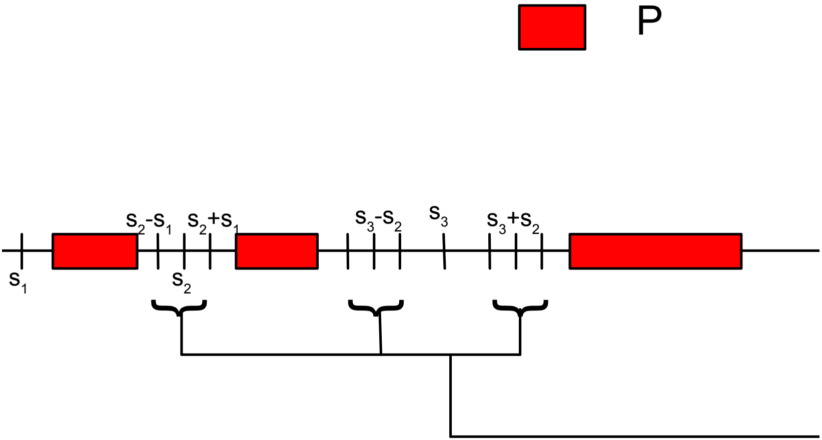
Suppose P is a set such that it does not have bounded gaps. Then we can fit an SIP set in its complement.



Predictive sets are $SIP^\star$

Theorem (Chandgotia, Weiss)

Predictive sets are $SIP^\star$.

1. If $r \not\equiv 0 \pmod{k}$ then $k\mathbb{N} + r$ is not SIP^*: Thus we have generalised the fact that $k\mathbb{N} + r$ is not predictive.
2. $SIP^\star$ sets have bounded gaps. Thus predictive sets also have bounded gaps.

Sufficient conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Return-time sets are predictive sets.

Necessary conditions for a set to be predictive:

Theorem (Chandgotia, Weiss)

Predictive sets are SIP*.

The following question arises naturally.

Question

Are sufficient conditions necessary and necessary conditions sufficient?

Let us give some partial answers.

Are all $SIP^\star$ sets predictive?

If $\epsilon > 0$ and $\alpha \in \mathbb{R}/\mathbb{Z}$ then

$$\{n \in \mathbb{N} \ : \ n\alpha \in (-\epsilon, \epsilon)\}$$

is predictive.

Are all $SIP^\star$ sets predictive?

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Is the intersection of two predictive sets also predictive? Is the intersection non-empty?

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Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be irrational and $\epsilon < 1/2$. Is the set

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predictive?

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predictive?

If the answer is yes then we have two predictive sets

$$\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\} \text{ and } \{n \in \mathbb{N} \ : \ -n\alpha \in (0, \epsilon)\}$$

which do not intersect.

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which do not intersect.

Theorem (Akin and Glasner, 2016)

The set $\{n \in \mathbb{N} \ : \ n\alpha \in (0, \epsilon)\}$ is $SIP^\star$.

Thus if the answer is no then we have a $SIP^\star$ set which is not predictive.

So we don't really know if all $SIP^\star$ sets are predictive.

There are predictive sets which do not contain return-time sets.

Consider the set

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then since i and $-1 + 3ik$ are prime to each other, they are perfect squares themselves.

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But this is impossible because $-1 + 3ik \equiv -1 \pmod 3$. Thus $\mathbb{N} \setminus Q$ contains $-i + 3i^2k; k \in \mathbb{N}$.

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Hence we have that

$$H(X_{-i} \mid X_{\mathbb{N} \setminus Q}) = 0$$

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But then for all $i \in \mathbb{Z}$

$$H(X_i \mid X_{\mathbb{N}\setminus Q}) = H(X_i \mid X_{(-\mathbb{N})\cup(\mathbb{N}\setminus Q)}) = 0.$$

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But all return-time sets must intersect the set $\{n^2 \ : \ n \in \mathbb{N}\}$ (Sarkozy, Furstenberg). Thus there are predictive sets which are not return-time sets.

Predictive sets

Question

Let $\{n_k\}_{k\in\mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that

$$H(X_0 \mid X_{\mathbb{N}\setminus\{n_k \mid k\in\mathbb{N}\}}) = 0.$$

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We do not know this even in the case $n_k = k^3$.

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We do not know this even in the case $n_k = k^3$. The only partial progress we have made towards this question uses the Fermat's last theorem.

Proofs.

Return-time sets are predictive

Given a process $Y_{\mathbb{Z}}$ with U in its state space, we write

$$N(U, U) := \{n \in \mathbb{N} \ : \ \mathrm{Prob}(Y_0 = Y_n = U) > 0\}.$$

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It is sufficient to prove that the difference set of a positive density set is predictive.

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

$$\alpha = \lim_{n\to\infty} \frac{n}{q_n} > 0$$

and $h(X_{\mathbb{Z}}) = 0$.

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and $h(X_{\mathbb{Z}}) = 0$.Then

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \ldots, X_{q_n}) \leq \frac{q_n}{n} \frac{1}{q_n} H(X_1, X_2, \ldots, X_{q_n}) \to \frac{1}{\alpha} h(X_{\mathbb{Z}}) = 0.$$

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But

$$\frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) \quad = \quad \frac{1}{n} H(X_0 \mid X_{q_2-q_1}, X_{q_3-q_1}, \dots, X_{q_n-q_1})$$

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But

$$\begin{aligned} \frac{1}{n} H(X_{q_1}, X_{q_2}, \dots, X_{q_n}) &= \frac{1}{n} H(X_0 \mid X_{q_2-q_1}, X_{q_3-q_1}, \dots, X_{q_n-q_1}) \\ &+ \frac{1}{n} H(X_0 \mid X_{q_3-q_2}, X_{q_4-q_2}, \dots, X_{q_n-q_2}) + \dots \end{aligned}$$

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&+ \frac{1}{n} H(X_0 \mid X_{q_n-q_{n-1}})
\end{aligned}$$

Return-time sets are predictive

Let $Q = \{q_1 < q_2 < q_3 < \dots\}$ have density

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&\geq H(X_0 \mid X_{(Q-Q)\cap\mathbb{N}})
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Return-time sets are predictive

Thus if Q has positive density then

$$H(X_0 \mid X_{(Q-Q)\cap\mathbb{N}}) = 0$$

and $(Q-Q)\cap\mathbb{N}$ is a predictive set. We showed earlier that every return-time set contains such a set.

Thus return-time sets are predictive.

Predictive sets are $SIP^\star$

In course of the proof we show that for all $SIP(S)$ there exists a weak mixing zero entropy Gaussian process $X_{\mathbb{Z}}$ such that

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Thus there exists a weak-mixing process in which X_0 can be predicted by $X_{\mathbb{N}}$ but is independent of $X_{2\mathbb{N}+1}$.

Predictive sets are $SIP^{\star}$: Processes and Spectral measures

From here on we will assume that X_0 is complex-valued, has zero mean and finite variance.

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On the other hand, given any finite positive measure μ on $\mathbb{R}/\mathbb{Z}$ there exists a Gaussian process $X_{\mathbb{Z}}$ such that

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$$\begin{aligned} f_r(x) &:= \prod_{k \le r} (1 + \cos(2\pi s_k x)) \\ &= \prod_{k \le r} \left(1 + \frac{\exp(2\pi i s_k x) + \exp(-2\pi i s_k x)}{2} \right). \end{aligned}$$

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As r tends to infinity the limit of $f_r \mu_{Leb}$ is a singular continuous measure μ such that $\hat{\mu}(n) = 0$ for all

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Thus $X_{\mathbb{Z}}$ has zero entropy, is weak mixing and $\mathbb{E}(X_0 \overline{X_n}) = 0$ for all

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If P is predictive then

$$P \cap SIP(s_1, s_2, \ldots) \neq \emptyset.$$

One can use this to prove that predictive sets are SIP^{*}.

Totally Predictive sets

A set P is a totally predictive set if it can predict everything, that is, for all zero entropy processes $X_{\mathbb{Z}}$ and $n \in \mathbb{N}$,

$$H(X_{-n} \mid X_P) = 0.$$

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We can prove the following using all of this machinery.

Theorem (Chandgotia, Weiss)

Let P be a totally predictive set and μ be any (complex-valued finite measure) on $\mathbb{R}/\mathbb{Z}$ such that the support of $\hat{\mu}$ is on $\mathbb{Z} \setminus P$. Then μ must have an absolutely continuous component.

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This is very close to Riesz sets as defined by Yves Meyer in 1968: A set $Q \subset \mathbb{Z}$ is called a Riesz set if all measures on $\mathbb{R}/\mathbb{Z}$ whose Fourier coefficients are supported on Q are absolutely continuous.

Riesz Sets

A set $P \subset \mathbb{N}$ is called totally predictive if $P + i$ is predictive for all $i \in \mathbb{N}$.

Theorem (Chandgotia, Weiss)

If $P \subset \mathbb{N}$ is a totally predictive set which is open in the Bohr topology, then $\mathbb{Z} \setminus P$ is a Riesz set.

Question

If $P \subset \mathbb{N}$ is totally predictive then is $\mathbb{Z} \setminus P$ a Riesz set? If $Q \subset \mathbb{N}$ is a set such that $Q \cup (-\mathbb{N})$ is Riesz then is $\mathbb{N} \setminus Q$ a totally predictive set?

A titillating question

Let $n_{\mathbb{N}}$ be an increasing sequence of natural numbers such that $n_{i+1} - n_i$ is also an increasing sequence. We had asked whether $\mathbb{N} \setminus n_{\mathbb{N}}$ is totally predictive.

It is unknown even for $n_i = i^3$ whether $(-\mathbb{N}) \cup n_{\mathbb{N}}$ is a Riesz set. Wallen (1970) proved that if μ is a measure whose Fourier coefficients are supported on $(-\mathbb{N}) \cup n_{\mathbb{N}}$ then $\mu \star \mu$ is absolutely continuous.

Following an idea by Lindenstrauss, a simple application of Fermat's last theorem and Cauchy Schwarz gives us the following partial result.

Theorem (Chandgotia, Weiss)

If μ is a probability measure whose Fourier coefficients are supported on $\{\pm i^K \ : \ i \in \mathbb{N}\} \cup \{0\}$ for some $k \geq 2$ then μ is not singular.

$k \geq 3$ is odd and $Q = \{n^k \ : \ n \in \mathbb{Z}\}$

Theorem (Chandgotia, Weiss, 2020)

If μ is a probability measure such that Support$(\hat{\mu}) \subset Q$ then μ is absolutely continuous.

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By Fermat's last theorem $Q^+ + Q^+$ is disjoint from Q^+, the right hand side is equal to $\sum_{j \in Q^+} |a_j|^2$. Let $Q_n := Q \cap [1, n]$. Applying the inequality thus obtained to the polynomial $c(z) := \sum_{j \in Q_n} \hat{\mu}(j) z^j$ we get that

$$\sum_{j \in Q_n} |\hat{\mu}(j)|^2 \leq 1$$

for all $n \in \mathbb{N}$. This proves that μ is absolutely continuous. $\square$

Summary

Return-time sets are predictive.

The converse is not true.

Predictive sets are $SIP^{\star}$.

Predictive sets have bounded gaps.

Some questions

1. Suppose a set P is predictive. Can P predict any thing in the future? In other words, is there $i \in -\mathbb{N}$ such that $H(X_i \mid X_P) = 0$ for all zero entropy processes $X_{\mathbb{Z}}$?
2. Is the intersection of two predictive sets also a predictive set?
3. Are all $SIP^\star$ sets predictive?
4. Is $\{n \ : \ n\alpha \in (0, \epsilon)\}$ a predictive set for irrational α?
5. Let $\{n_k\}_{k \in \mathbb{N}}$ be an increasing sequence such that $n_{k+1} - n_k$ is also an increasing sequence. Prove that for all zero entropy processes $X_{\mathbb{Z}}$,

$$H(X_0 \mid X_{\mathbb{N} \setminus \{n_k \mid k \in \mathbb{N}\}}) = 0.$$

6. What is the relationship between Riesz sets and totally predictive sets?