Markov Random Fields and the 3-coloured Chessboard

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Outline

- Topological Markov fields
- Markov random fields and Gibbs measures with nearest neighbour interactions
- The pivot property
- Examples: 3-coloured chessboard and the Square Island shift.

Topological Markov Fields

A topological Markov field is a shift space $X \subset \mathcal{A}^{\mathbb{Z}^d}$ with the 'conditional independence' property: for all finite subsets $F \subset \mathbb{Z}^d$, $x, y \in X$ satisfying $x = y$ on ∂F, $z \in \mathcal{A}^{\mathbb{Z}^d}$ given by

$$z = \begin{cases} x \text{ on } F \\ y \text{ on } F^c \end{cases}$$

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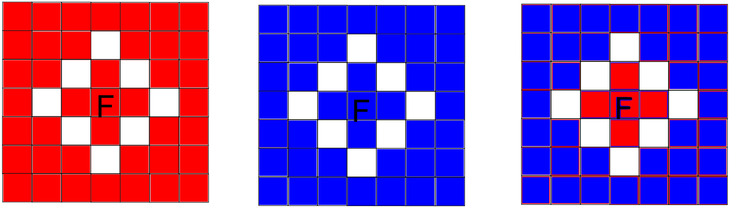

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Consider any one-dimensional shift space X. Make a two-dimensional shift space Y where the horizontal constraints come from X and the vertical direction is constant. If x and y agree on ∂F, they must agree on F. Therefore Y is a topological Markov field. There are uncountably many such shift spaces but there are only countably many nearest neighbour shift of finite type!!!

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A Gibbs state with a nearest neighbor interaction V is a Markov random field μ such that for all $x \in supp(\mu)$ and $A, B \subset \mathbb{Z}^d$ finite satisfying $\partial A \subset B \subset A^c$

$$\mu([x]_A \,\Big|\, [x]_B) \;=\; \frac{\prod_{C \subset A \cup \partial A} e^{V([x]_C)}}{Z_{A, x|_{\partial A}}}$$

where $Z_{A, x|_{\partial A}}$ is the uniquely determined normalising factor so that $\mu(X) = 1$, dependent upon A and $x|_{\partial A}$.

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The specification of a Gibbs measure with a nearest neighbour interaction has a finite description: all we need is the interaction V.

Question: When is a Markov random field Gibbs with some nearest neighbour interaction?

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Question: How can we weaken the hypothesis?

Pivot Property

A shift space X is said to satisfy the pivot property if for all $x, y \in X$ which differ only on finitely many sites there exists a chain $x = x^1, x^2, x^3, \ldots, x^n = y \in X$ such that x^i, x^{i+1} differ on at most a single site.

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Examples:

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Examples:

- Any shift space with a safe symbol.
- r-coloured checkerboard for $r \neq 4, 5$.
- Domino tilings.

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Thus the space of specifications on any topological Markov field with the pivot property can be parametrised by finitely many parameters.

Question: Suppose we are given a nearest neighbour shift of finite type with the pivot property. Is there an algorithm to determine the number of parameters which describes the specification?

A specification supported on the 3-coloured chessboard is determined the quantities $v_1 = \dfrac{\mu(\begin{bmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{bmatrix})}{\mu(\begin{bmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{bmatrix})}$, A specification supported on the 3-coloured chessboard is determined the quantities $v_1 = \dfrac{\mu(\begin{bmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{bmatrix})}{\mu(\begin{bmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{bmatrix})}$, $v_2 = \dfrac{\mu(\begin{bmatrix} & 2 & \\ 2 & 1 & 2 \\ & 2 & \end{bmatrix})}{\mu(\begin{bmatrix} & 2 & \\ 2 & 0 & 2 \\ & 2 & \end{bmatrix})}$

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$$\begin{aligned}
v_1 &= exp(V(01) + V(10) + V(\substack{0\\1}) + V(\substack{0\\1}) \\
&\quad -V(21) - V(12) - V(\substack{2\\1}) - V(\substack{1\\2})), \\
v_2 &= exp(V(12) + V(21) + V(\substack{2\\1}) + V(\substack{1\\2}) \\
&\quad -V(02) - V(20) - V(\substack{0\\2}) - V(\substack{2\\0})), \\
v_3 &= exp(V(02) + V(20) + V(\substack{2\\0}) + V(\substack{0\\2}) \\
&\quad -V(01) - V(10) - V(\substack{0\\1}) - V(\substack{1\\0})).
\end{aligned}$$

μ is Gibbs if and only if $v_1 v_2 v_3 = 1$.

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Thus the Hammersley-Clifford type conclusion holds for fully supported measures.

What if the pivot property does not hold? Every 1 dimensional nearest neighbour shift of finite type has the generalised pivot property.

Square Island Shift

Inspiration from checkerboard island shift by Quas and Şahin.

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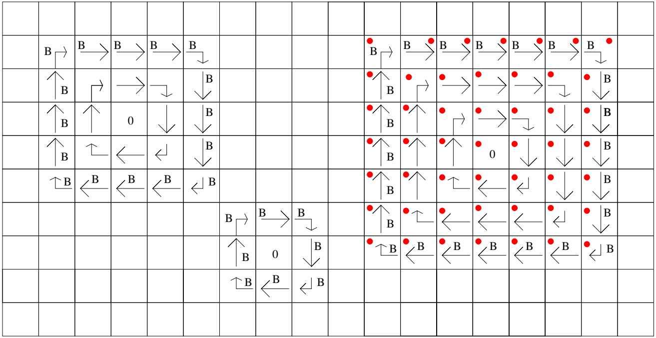

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There are two kinds of squares: ones with red dots and ones without red dots which float in a sea of blanks.

The Square Island shift does not have the generalised pivot property.

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There is no way to switch from a big square with red dots to a big square without red dots making single site changes(or even bigger regional changes).

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Question: Can more uniform mixing conditions help?

Thank You!

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