Graph Foldings and Markov Random Fields

Nishant Chandgotia

University of British Columbia

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Outline

- Markov random fields and Gibbs measures with nearest neighbour interactions
- Review of previous results
- The pivot property
- Graph folding and Hammersly-Clifford spaces

Some Notation and Setting

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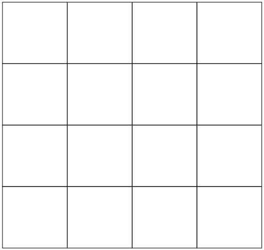

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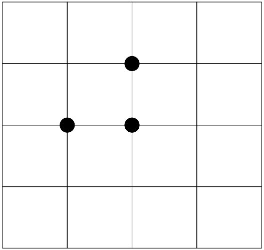

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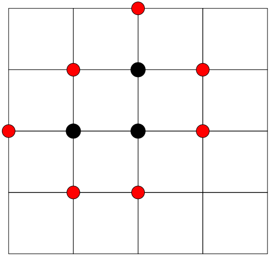

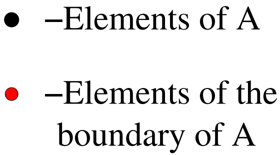

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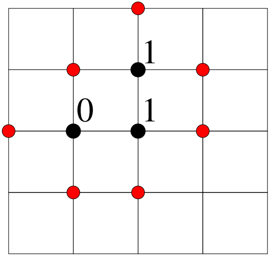

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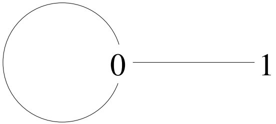

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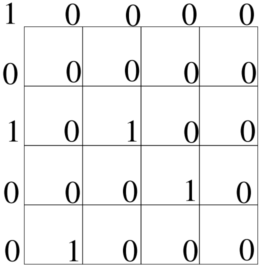

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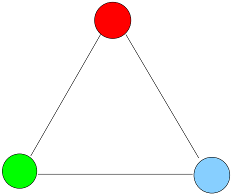

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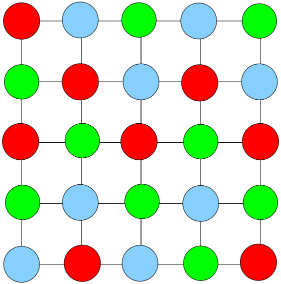

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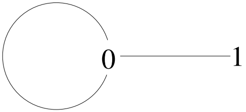

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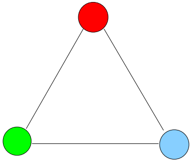

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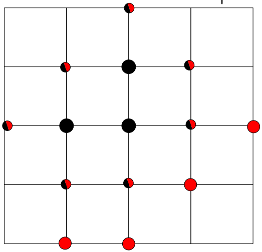

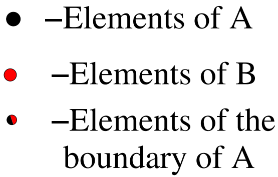

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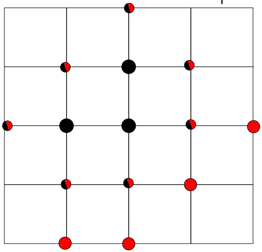

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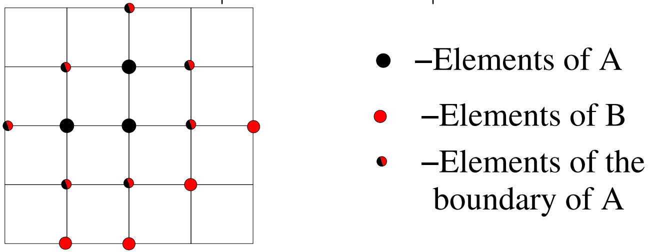

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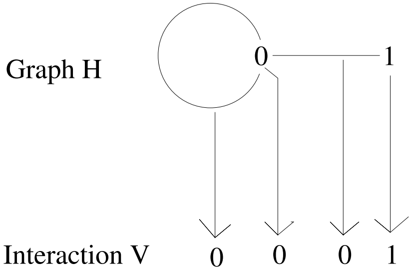

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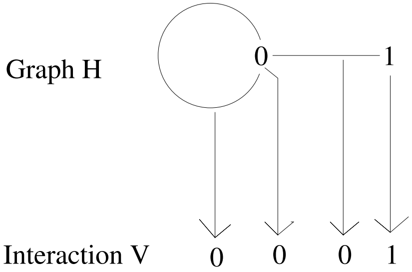

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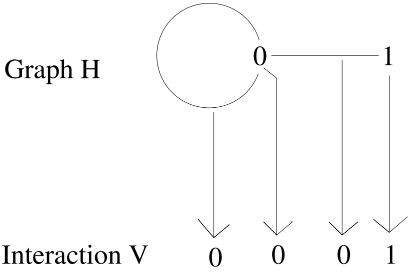

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Question: Under what conditions on the support is a Markov random field Gibbs with some nearest neighbour interaction?

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- For shift-invariant measures and $\mathcal{G} = \mathbb{Z}$: Chandgotia, Han, Marcus, Meyerovitch and Pavlov('11)
- For shift-invariant measures and support $Hom(\mathbb{Z}^d, \mathcal{H})$ where $\mathcal{H}$ is an n cycle($n \neq 4$): Chandgotia and Meyerovitch('13)

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The measure we obtained was not Gibbs for any finite range interaction.

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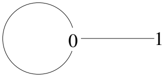
- When $\mathcal{G} = \mathbb{Z}$ and the measure is not shift-invariant: Dobruschin('68)
- When the alphabet is countable: Georgii('88)
- A shift-invariant measure when $\mathcal{G} = \mathbb{Z}^2$: Chandgotia and Meyerovitch('13)

Remark:
The measure we obtained was not Gibbs for any finite range interaction.
The support of the measure cannot be represented as $Hom(\mathbb{Z}^2, \mathcal{H})$ for any graph $\mathcal{H}$.

Consequences of having a Safe Symbol

Suppose $Hom(\mathcal{G}, \mathcal{H})$ is the hard square model.



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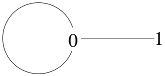

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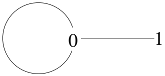

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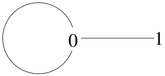

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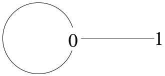

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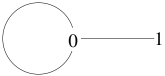

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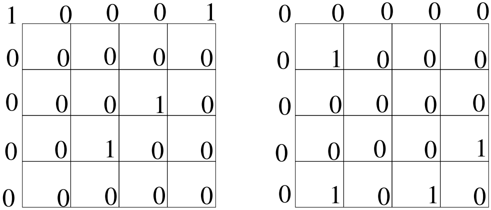

x y

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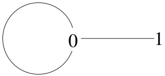

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$$\begin{array}{ccccc} \color{red}{1} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}$$

x^1 $\qquad$ y

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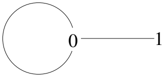

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0	0	0	0	1
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	0	0	0

x^2

0	0	0	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	1
0	1	0	1	0

y

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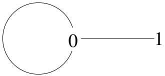

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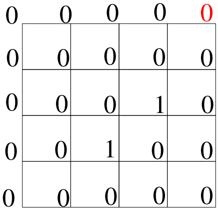
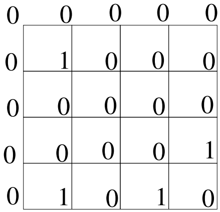


x^3



y

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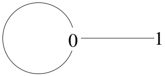

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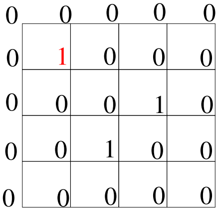
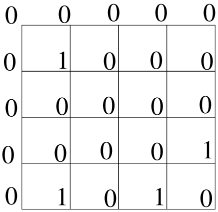


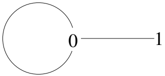
x^4



y

Consequences of having a Safe Symbol

Suppose $Hom(\mathcal{G},\mathcal{H})$ is the hard square model.



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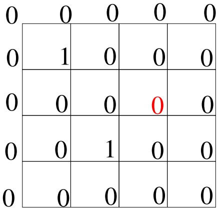
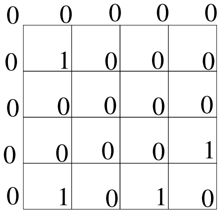


x^5



y

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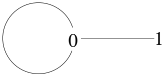

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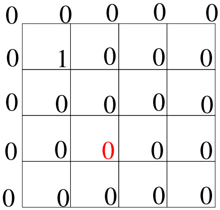
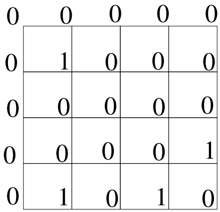


x^6



y

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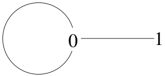

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0	0	0	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	1
0	0	0	0	0

x^7

0	0	0	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	1
0	1	0	1	0

y

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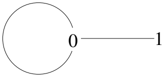

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0	0	0	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	1
0	1	0	0	0

x^8

0	0	0	0	0
0	1	0	0	0
0	0	0	0	0
0	0	0	0	1
0	1	0	1	0

y

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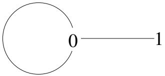

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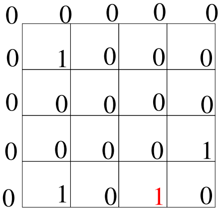
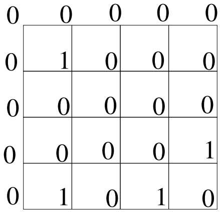


x^9



y

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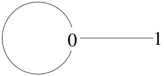

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$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array} \qquad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{array}$$

y y

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- If $Hom(\mathcal{G}, \mathcal{H})$ has a safe symbol.
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- If $\mathcal{H}$ is dismantleable (to be defined in the next few slides).

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$$\frac{\mu([x]_F \mid [x]_{\partial F})}{\mu([y]_F \mid [x]_{\partial F})} = \prod_{i=1}^{n-1} \frac{\mu([x^i]_F \mid [x^i]_{\partial F})}{\mu([x^{i+1}]_F \mid [x^i]_{\partial F})}$$

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$$\begin{aligned}
\frac{\mu([x]_F \mid [x]_{\partial F})}{\mu([y]_F \mid [x]_{\partial F})} &= \prod_{i=1}^{n-1} \frac{\mu([x^i]_F \mid [x^i]_{\partial F})}{\mu([x^{i+1}]_F \mid [x^i]_{\partial F})} \\
&= \prod_{i=1}^{n-1} \frac{\mu([x^i]_{m_i} \mid [x^i]_{\partial m_i})}{\mu([x^{i+1}]_{m_i} \mid [x^i]_{\partial m_i})}.
\end{aligned}$$

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\end{aligned}$$

Since μ is shift-invariant therefore the entire specification is determined by finitely many parameters viz. $\frac{\mu([x]_{0\cup\partial 0})}{\mu([y]_{0\cup\partial 0})}$ for configurations x, y which differ only at 0, the origin.

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Then ratios of the form $\frac{\mu([x]_{0\cup\partial 0})}{\mu([y]_{0\cup\partial 0})}$

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Then ratios of the form $\frac{\mu([x]_{0\cup\partial 0})}{\mu([y]_{0\cup\partial 0})}$ where x and y differ exactly on the origin

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Then ratios of the form $\frac{\mu([x]_{0 \cup \partial 0})}{\mu([y]_{0 \cup \partial 0})}$ where x and y differ exactly on the origin determine whether μ is Gibbs or not.

3-colourings of $\mathbb{Z}^2$

Let $\mathcal{H}$ be a 3-cycle with vertices 0, 1 and 2.

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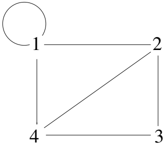

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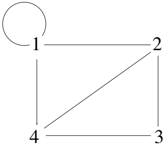

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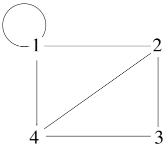

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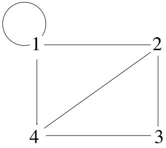

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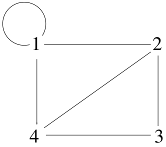

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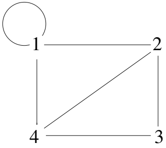

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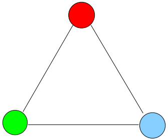

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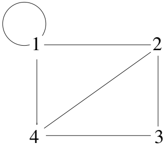

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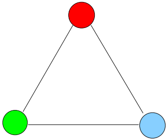

No vertex can be folded into the other. Such a graph is said to be stiff.

Hammersley-Clifford Spaces

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Graph Folding and Hammersley-Clifford Spaces

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Let $\mathcal{G}$ be bipartite graph and $\mathcal{H}$ be a graph with a fold $\mathcal{H} \setminus \{v\}$. Then the space $Hom(\mathcal{G}, \mathcal{H})$ is a Hammersley-Clifford space if and only if $Hom(\mathcal{G}, \mathcal{H} \setminus \{v\})$ is a Hammersley-Clifford space as well.

Remark:

- There is a corresponding version of the theorem where the specifications and interactions are assumed to be invariant under some automorphism of the graph $\mathcal{G}$ e.g. translations in the case of $\mathbb{Z}^d$.

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- There is a corresponding version of the theorem where the specifications and interactions are assumed to be invariant under some automorphism of the graph $\mathcal{G}$ e.g. translations in the case of $\mathbb{Z}^d$.
- This result is true for a more general notion of folding on closed spaces of configurations, not just restricted to homomorphism spaces.

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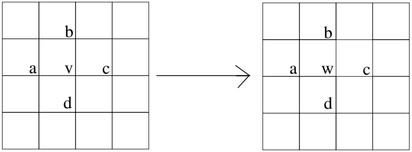

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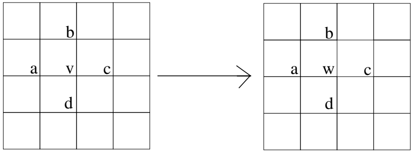

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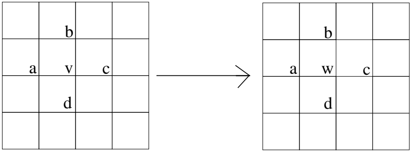

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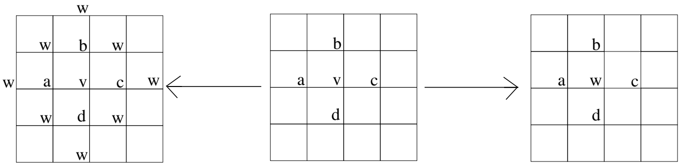

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Since $Hom(\mathbb{Z}^2, \mathcal{H} \setminus \{v\})$ is a Hammersley-Clifford space we only care about pairs which involve changing a single v to w.

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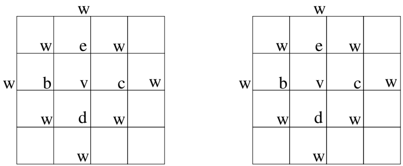

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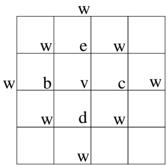

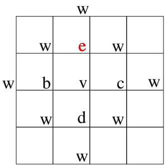

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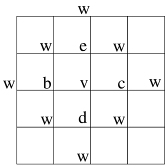

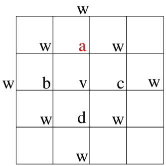

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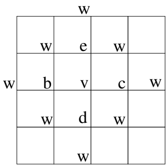

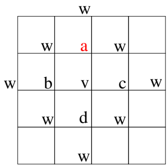

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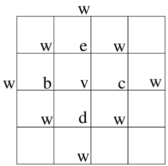

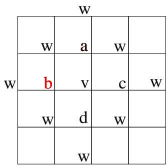

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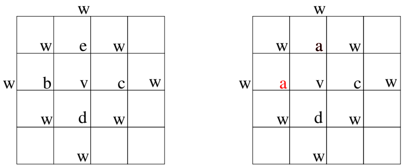
We can use the following order to change a single v to w:



An idea of the proof:

Assume $v \not\sim v$ and choose some $a \sim v$. For all $x \sim v$ make the following identifications

$$
\begin{array}{lll}
V([xv]) & \text{with} & w \begin{matrix} w \\ x \\ w \end{matrix} v \, , \; w \begin{matrix} w \\ a \\ w \end{matrix} v \\
V([vx]) & \text{with} & \color{red}{v \begin{matrix} w \\ x \\ w \end{matrix} w \, , \; v \begin{matrix} w \\ a \\ w \end{matrix} w} \\
V\left(\left[\begin{smallmatrix} v \\ x \end{smallmatrix}\right]\right) & \text{with} & w \begin{matrix} v \\ x \\ w \end{matrix} w \, , \; w \begin{matrix} v \\ a \\ w \end{matrix} w \\
V\left(\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]\right) & \text{with} & w \begin{matrix} w \\ x \\ v \end{matrix} w \, , \; w \begin{matrix} w \\ a \\ v \end{matrix} w \\
V([v]) & \text{with} & a \begin{matrix} a \\ v \\ a \end{matrix} a \, , \; a \begin{matrix} a \\ w \\ a \end{matrix} a
\end{array}
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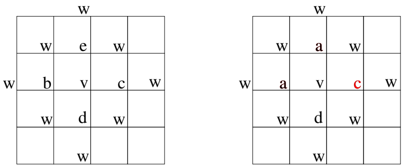

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V([vx]) & \text{with} & \begin{array}{ccc} & w & \\ v & x & w \\ & w & \end{array}, \begin{array}{ccc} & w & \\ v & a & w \\ & w & \end{array} \\
V(\left[\begin{smallmatrix} v \\ x \end{smallmatrix}\right]) & \text{with} & \begin{array}{ccc} & v & \\ w & x & w \\ & w & \end{array}, \begin{array}{ccc} & v & \\ w & a & w \\ & w & \end{array} \\
V(\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]) & \text{with} & \begin{array}{ccc} & w & \\ w & x & w \\ & v & \end{array}, \begin{array}{ccc} & w & \\ w & a & w \\ & v & \end{array} \\
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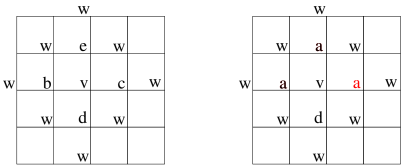

An idea of the proof:

Assume $v \nsim v$ and choose some $a \sim v$. For all $x \sim v$ make the following identifications

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V([xv]) & \text{with} & \begin{smallmatrix} & w & \\ w & x & v \\ & w & \end{smallmatrix}\ ,\ \begin{smallmatrix} & w & \\ w & a & v \\ & w & \end{smallmatrix} \\
V([vx]) & \text{with} & \begin{smallmatrix} & w & \\ v & x & w \\ & w & \end{smallmatrix}\ ,\ \begin{smallmatrix} & w & \\ v & a & w \\ & w & \end{smallmatrix} \\
V(\left[\begin{smallmatrix} v \\ x \end{smallmatrix}\right]) & \text{with} & \begin{smallmatrix} & v & \\ w & x & w \\ & w & \end{smallmatrix}\ ,\ \begin{smallmatrix} & v & \\ w & a & w \\ & w & \end{smallmatrix} \\
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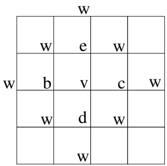

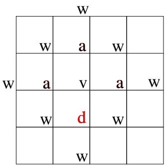

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Assume $v \nsim v$ and choose some $a \sim v$. For all $x \sim v$ make the following identifications

$$V([xv]) \quad \text{with} \quad \begin{array}{ccc} & w & \\ w & x & v \\ & w & \end{array}, \begin{array}{ccc} & w & \\ w & a & v \\ & w & \end{array}$$

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$$V\left(\left[\begin{smallmatrix} v \\ x \end{smallmatrix}\right]\right) \quad \text{with} \quad \color{red}{\begin{array}{ccc} & v & \\ w & x & w \\ & w & \end{array}, \begin{array}{ccc} & v & \\ w & a & w \\ & w & \end{array}}$$

$$V\left(\left[\begin{smallmatrix} x \\ v \end{smallmatrix}\right]\right) \quad \text{with} \quad \begin{array}{ccc} & w & \\ w & x & w \\ & v & \end{array}, \begin{array}{ccc} & w & \\ w & a & w \\ & v & \end{array}$$

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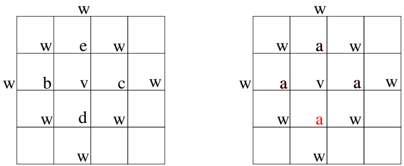

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Assume $v \not\sim v$ and choose some $a \sim v$. For all $x \sim v$ make the following identifications

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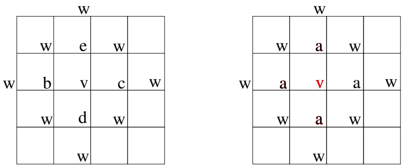

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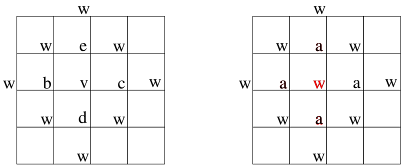

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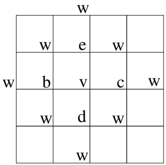

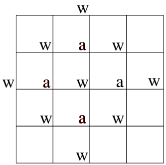

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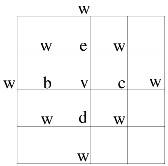

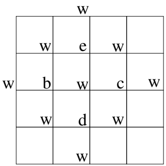

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Thus in the case when $v \not\approx v$ we can find an interaction which represents changing a single v to w in any configuration.

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If $v \sim v$ then the argument is slightly more involved.

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If $\mathcal{H}$ is a single vertex with a loop or an edge then $Hom(\mathcal{G}, \mathcal{H})$ is a Hammersley-Clifford space.

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If $\mathcal{H}$ is a single vertex with a loop or an edge then $Hom(\mathcal{G}, \mathcal{H})$ is a Hammersley-Clifford space.

Thus if $\mathcal{H}$ is dismantleable graph or a 4-cycle, then $Hom(\mathcal{G}, \mathcal{H})$ is a Hammersley-Clifford space.

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Thus if $\mathcal{H}$ is dismantleable graph or a 4-cycle, then $Hom(\mathcal{G}, \mathcal{H})$ is a Hammersley-Clifford space.

Are there any more such stiff graphs $\mathcal{H}$ in general?

Thank You!

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