# Some questions about tilings

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# Background & prerequisite

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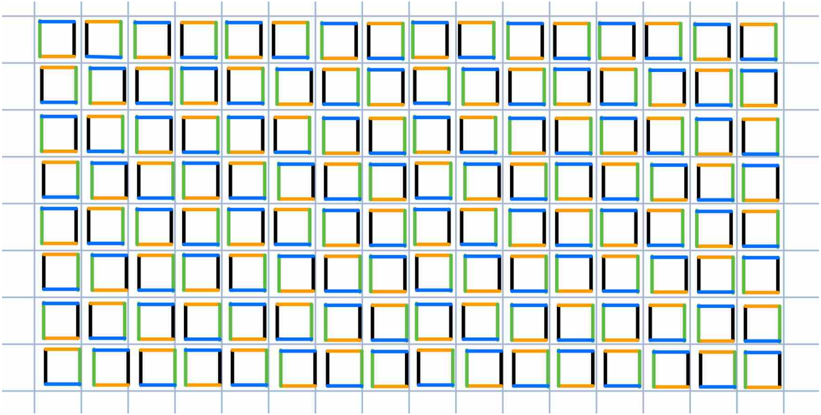
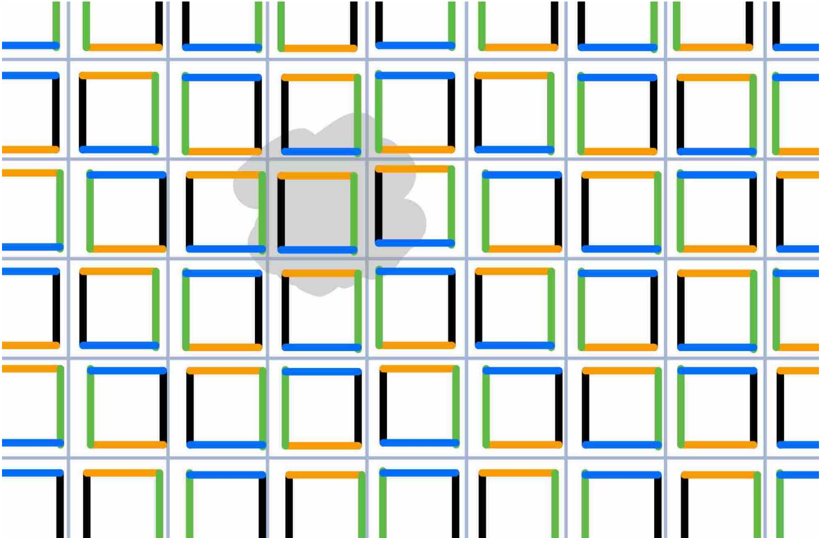


## Can you tile the plane?

But if the colour along an edge is not the same then you cannot.



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**Question**

*Can you cover the entire $\mathbb{Z}^2$ lattice with these tiles?*

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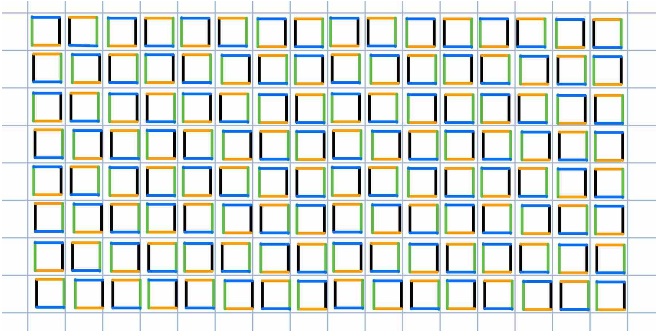
Of course! You can tile the plane periodically with these.



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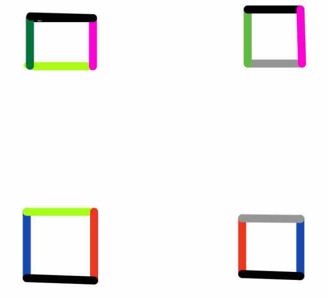
What about these tiles?



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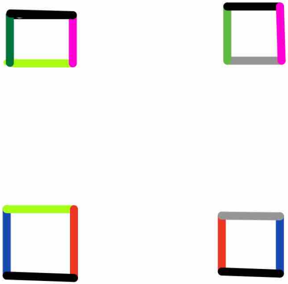
If at some point we see that we cannot tile any more then we know that no such tiling exists.



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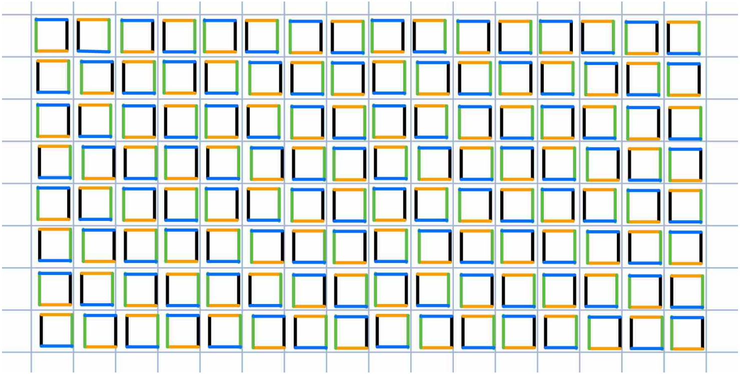

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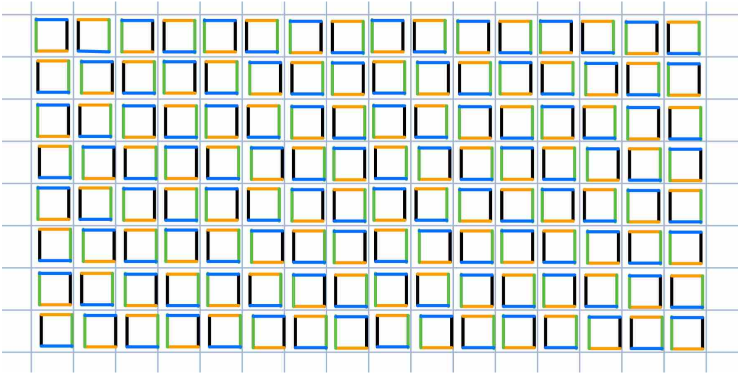

and get a tiling of the plane.

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But if in these attempts we see that the top and the bottom edge, the left and the right edge of some rectangle have the same colours then we continue periodically and get a tiling of the plane.

What is the problem with this strategy?

You may continue indefinitely and you might not encounter periodicity or non-tileability. This lead Hao Wang to ask the question:

Question (Wang, 1960)

*If there is a way to tile the plane, is there necessarily a way to tile the plane periodically?*

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What is the answer?

# Berger's undecidability theorem

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Alan Turing's thesis (1939) showed that there are provable things which can't be computed. Specifically he showed that there can be no algorithm to decide whether or not a given algorithm will halt.

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For every set of tiles it is either true or false whether it can tile the plane with it.

The theorem tells us that there can't be any general method which will work for every set of tiles.

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An infinite tiling will exist if and only if *'the computation will be endless'*.

Since there is no algorithm to decide the latter, there is no algorithm to decide the former either.

# Robinson's Undecidability and Nonperiodicity of Tilings of the Plane- 1971

Berger's example was complicated. Later Robinson gave an example which was much simpler.

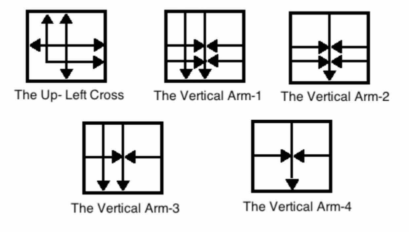


Figure: Robinson's tiles

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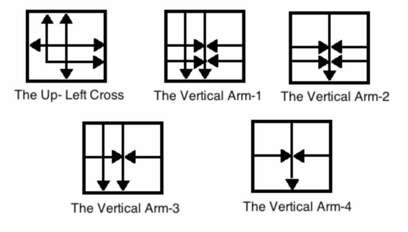
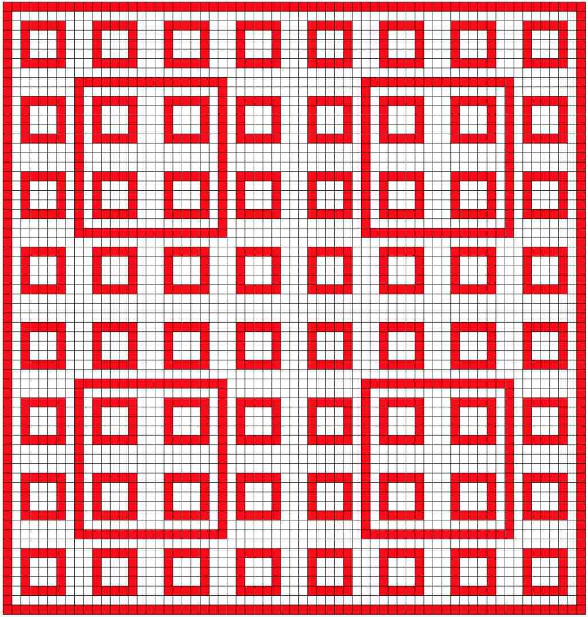
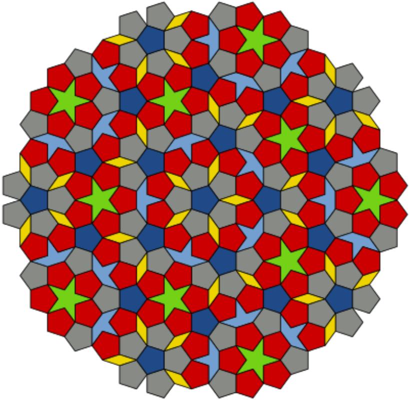



Figure: Robinson's tiles



In 1974, Roger Penrose came upon a set of tiles (no longer squares) which gave rise to special aperiodic tilings of the plane.



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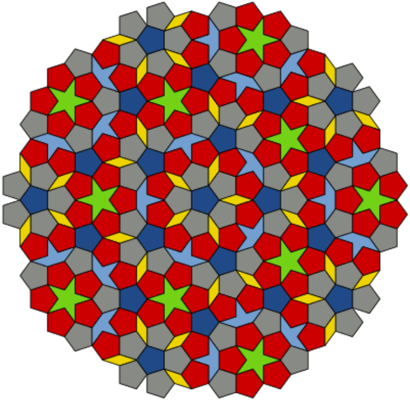

Note that it has some seeming 5-fold symmetry.

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In 1981 Steinhardt, Nelson and Ronchetti predicted that a "crystal" with such a structure must exist.

This prediction went against *crystallographic restriction theorem* which said that only $3, 4$ or $6$ fold rotational symmetry can exist in crystals.

However. . .

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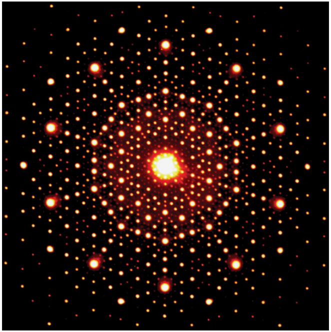

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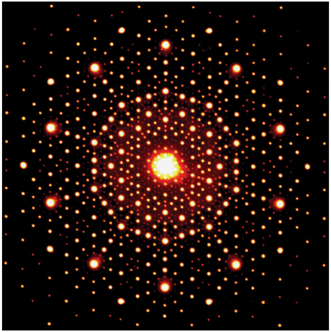

This broke the misconception that orderly arrangement of atoms must necessarily be periodic and brought a big revolution to the field of crystallography and material science.

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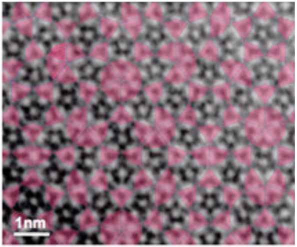
Dan Shechtman wins the Nobel prize in chemistry in 2011.

# Eiji Abe, 2012




An alloy under a electron microscope showing structure very similar to Penrose tilings.

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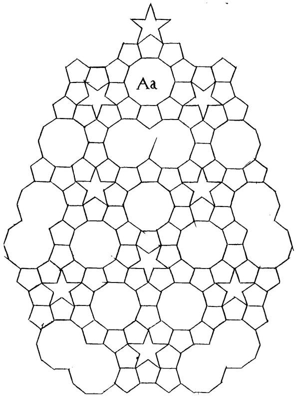

## Girih pattern from the Darb-i-Imam shrine, Iran. 1453 (Lu and Steinhardt)

We haven’t seen everything yet,
but when we do
it won’t be for the first time
or the last, either.
You know us.
-J. Vinograd

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Let us formalise some of these things.

# Shift of finite type

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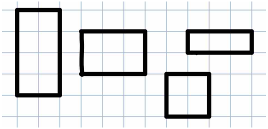
Let $A$ be a finite set and consider some rules $(R)$ how the symbols can be placed next to each other on the $\mathbb{Z}^d$ lattice.

The space of configurations on $\mathbb{Z}^d$ following the rules $(R)$ form what are called a shift of finite type.

# Example: Rectangular tiling shift

Suppose we have a bunch of boxes $T$.



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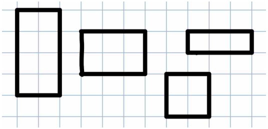

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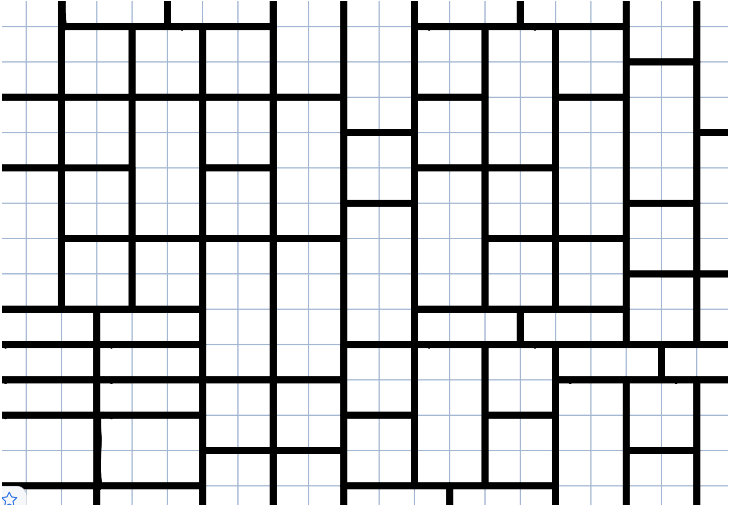
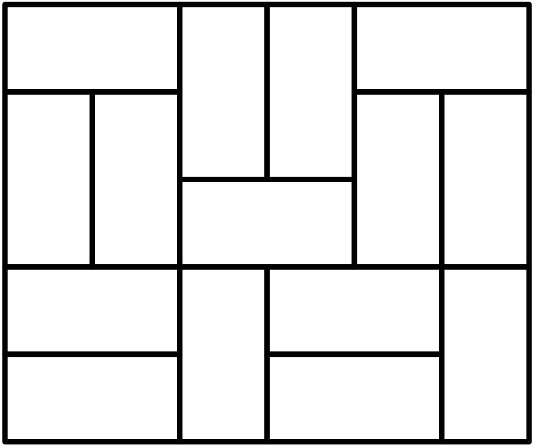
An extremely important subcase is that of domino tilings where all the sides have length 1 except one edge with length 2.



# Domino Tilings



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The entropy of the shift space associated with the rules $(R)$ is given by

$$h(R) = \lim_{n\to\infty} \frac{1}{n^d} \log(s_n(R)).$$

In 1961, Kastelyn computed the entropy of domino tilings in $d = 2$ (and later by Burton & Pemantle in 1993) as

$$\int_0^1 \int_0^1 \log\left(4 - 2\cos(2\pi\alpha_1) - 2\cos(2\pi\alpha_2)\right) d\alpha_1 d\alpha_2.$$

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It is very rare in this study to have such precise computations.

# Entropy can be approximated from above

Let $A$ be a finite set and consider some rules $(R)$ how the symbols can be placed next to each other on the $\mathbb{Z}^2$ lattice.

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So even though we cannot say whether the shift space is empty or not, we can always approximate the entropy from above.

# Right-recursively enumerable numbers

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**Theorem (Hochman-Meyerovitch)**

*A number $\beta = h(R)$ for some set of rules $R$ if and only if $\beta$ is non-negative and right-recursively enumerable.*

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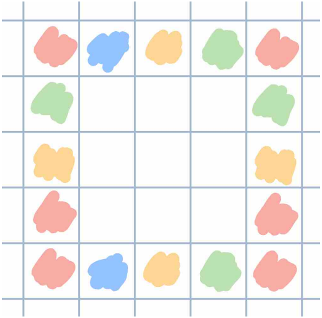
In "nice" cases an approximation from below is also possible.

For instance, if we have enough periodic points!!!

# With periodic points we can approximate entropy from below

Suppose $per_n(R)$ is the number of pattern on the $n \times n$ box with some fixed pattern on the top and bottom and on the left and the right.



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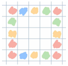
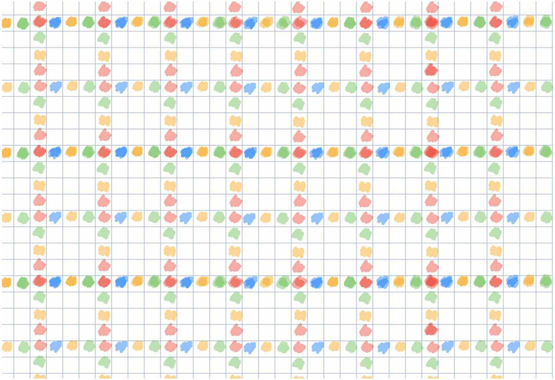


Then we can divide $\mathbb{Z}^2$ into $n \times n$ boxes each of which can be filled independently by an element of $per_n(R)$.



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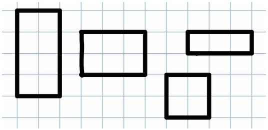
But even in the simplest case this is difficult to prove (or disprove).

Suppose we have a bunch of boxes $T$.



The only restriction is that both the *gcd* of the lengths and breadths is 1.

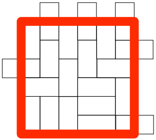
Look at all tilings by the boxes $T$.

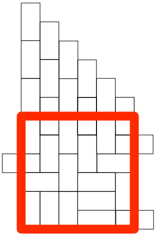
Question

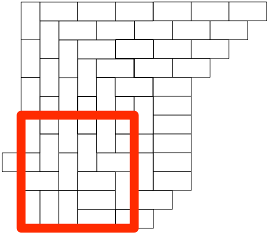
*Is the entropy of the space of tilings by these boxes computable?*

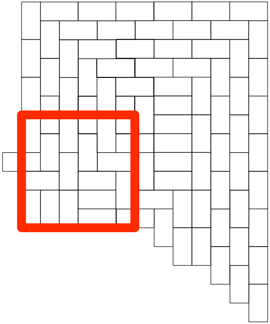
This question is wide open.

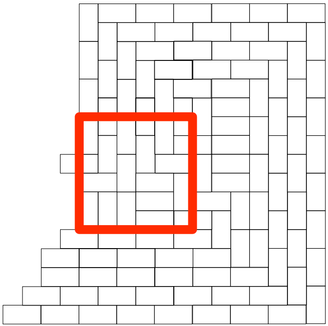
But domino tilings can be handled quite well.

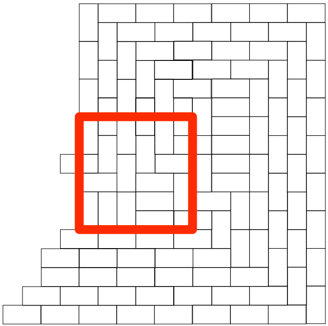


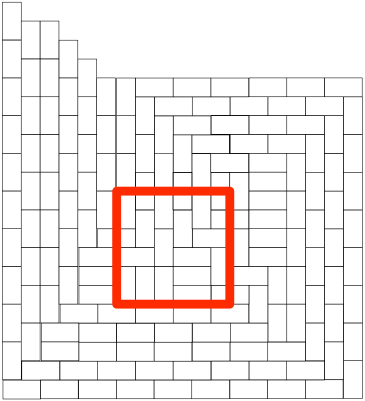


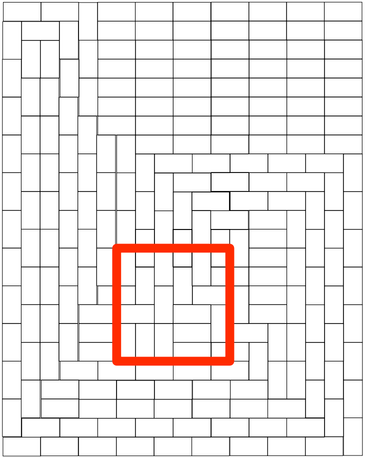


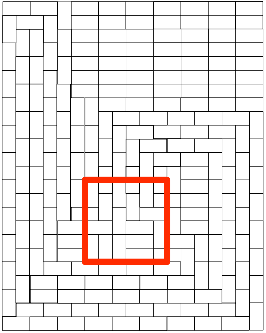












Theorem (Chandgotia)

*The entropy of domino tilings is computable in all dimensions.*

In $d = 2$ this follows from exact computations by Kastelyn(1961).

Here is a simpler question.

Question

*Can a partial tiling by rectangles always be completed to that of a box?*

For $d = 2$ this is known in the case of two tiles due to Einsedler (2001).

# Objective

We have three objectives:

1. Suppose we can tile the plane by a given set of tiles, can we count approximately how many of them are there?
2. What kind of processes can tilings model?
3. What does a random tiling actually look like?

# Modelling of $\mathbb{Z}^d$ actions

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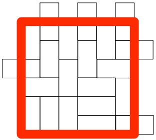

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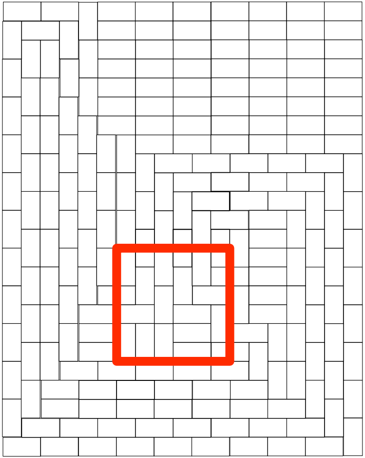

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A small caveat here is that we have to get rid of a set of "universal measure zero" from $X$. A lot of progress has been made to deal with this caveat in recent work with Spencer Unger.

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Following Kastelyn (1961) and Burton & Pemantle (1993) this follows easily for domino tilings in two dimensions.

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Theorem (Chandgotia)

*For all dimensions, if $T$ is the set of domino tiles then*

$$h(T) = \limsup_{n\to\infty} \frac{1}{n^d} \log(|Box_n(T)|).$$

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Again, we do not know anything about the general case beyond dominos.

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The general question here is the following: Suppose we are given a 'nice' subset of $\mathbb{Z}^d$. What does a random tiling of that region look like?

# Cohn, Kenyon & Propp, 2000

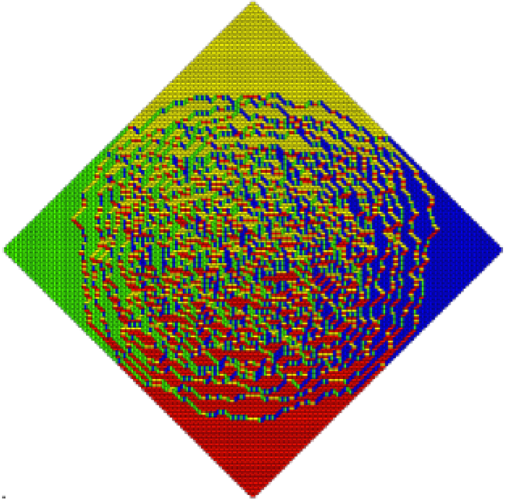

Figure: Simulation by Martin Tassy

# Domino tilings of the $\mathbb{Z}^3$ lattice

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We believe that everything extends to higher dimensions but this is the subject of ongoing work.

# The monotiling conjecture

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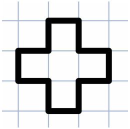

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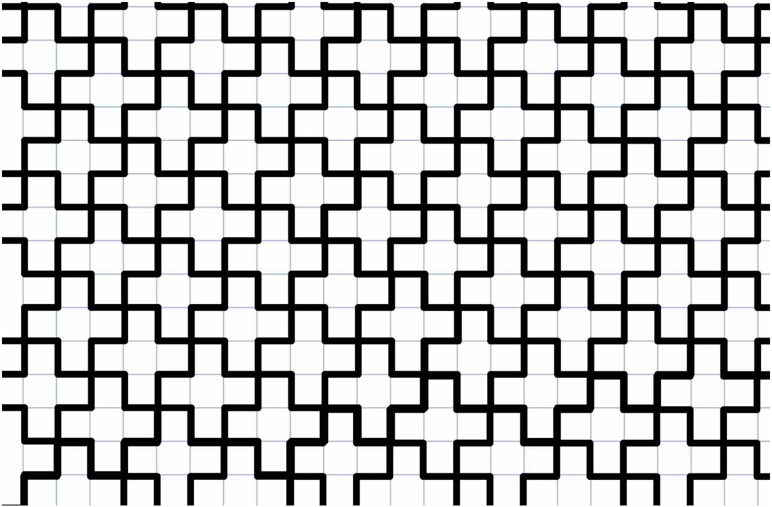

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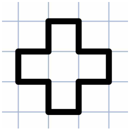

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It is wide open in $\mathbb{R}^2$ and $\mathbb{Z}^d$ for $d > 2$. There has been some recent progress by Abhishek Khetan (2021) for tilings of $\mathbb{Z}^3$.

Even for tilings of $\mathbb{Z}$ we do not have any understanding of what kind of tiles can tile $\mathbb{Z}$. This goes into some interesting number theoretic questions (Coven & Meyerowitz, 1998).

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Happy solving!!!