The Dimer Model in 3 dimensions

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Many thanks to Spencer Unger for hosting me in Toronto and Balint Virag and Benjamin Landon for the invitation for this talk.

This is joint work with Scott Sheffield and Catherine Wolfram.

All of the beautiful simulations and graphics have been made by Scott Sheffield and Catherine Wolfram.

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We will focus mostly on $d = 3$.

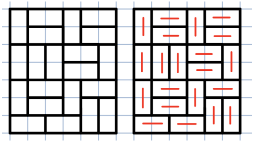

Figure : A dimer tiling on the left and a perfect matching on the right

When can a set be tiled?

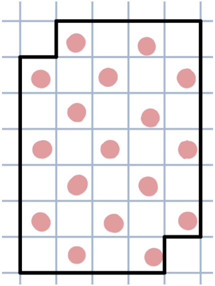

Figure : The red dots are the elements of P_1 and the undotted ones are the elements of P_2. A dimer tiling does not exist because $|P_1| > |P_2|$.

Suppose we want to find out whether a set $F \subset \mathbb{Z}^d$ can be perfectly matched. The set F can be divided into two partite classes P_1, P_2. Now if F can be perfectly matched, each vertex in P_1 is perfectly matched with each vertex in P_2 and vice versa. In particular $|P_1| = |P_2|$.

A criterion for finding a tiling: Hall's marriage lemma

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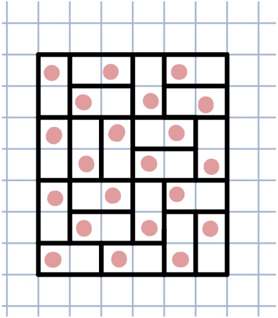

Figure : The red dots are the elements of P_1 and the undotted ones are the elements of P_2. It satisfies the criterion for perfect matching.

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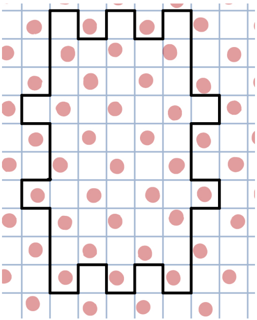

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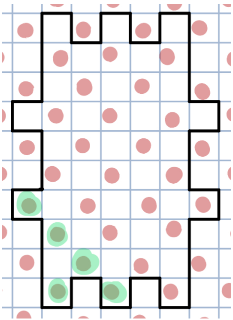
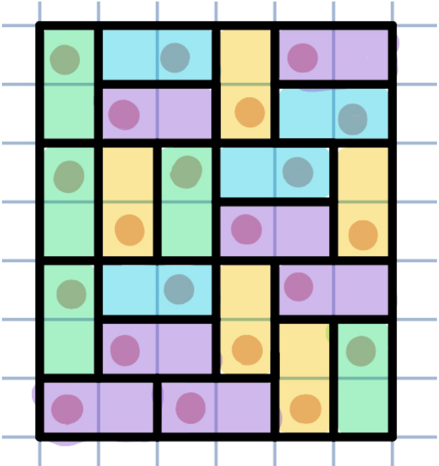
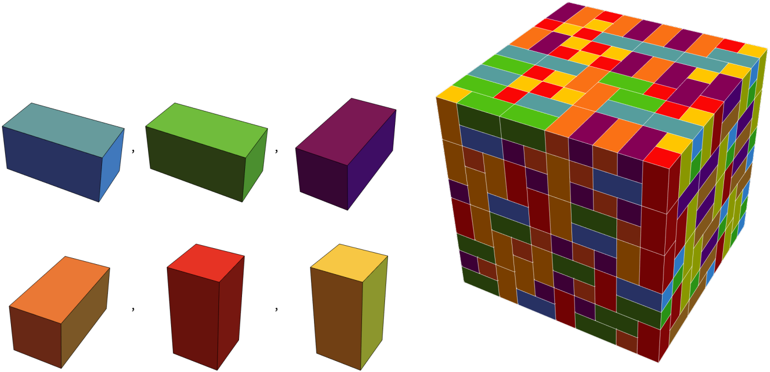


Figure : The red dots are the elements of P_1 and the undotted ones are the elements of P_2. It does not satisfy the criterion for perfect matchings. There are five elements of P_1 with the green shade with only four neighbours.

Parity is important

In general parity is important. Thus we will distinguish two different translates of the same domino but with different parity. $\mathbb{Z}^d$ will have $2d$ differ kinds of dominos.





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This is a wide ranging question and we are interested in all possible interpretations. However for this talk we will concentrate on a certain large deviations principle.

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Specifically, if N is the number of coins and M_N is the difference of the number of heads and tails then for $x > 0$

$$\mathbb{P}(M_N/N > x) \approx e^{-N\,I(x)}.$$

Here $I(x)$ is half the Shannon entropy.

Some Simulations

Let us see some simulations to get a feeling for what the "mean behaviour" of the domino tilings looks like.

Some questions

Take a contractible open set $R \subset \mathbb{R}^3$ and take a sequence of sets $R_n \subset \mathbb{Z}^3$ such that $\frac{1}{n} R_n$ approximates R in the Hausdorff topology. We want to look at uniformly sampled tilings on R_n and study its (possibly random/deterministic) limit as $n \to \infty$.

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But before we make this more rigorous we need to first understand in which space is this convergence happening.

For this it will be instructive to understand how things are formulated in 2 dimensions.

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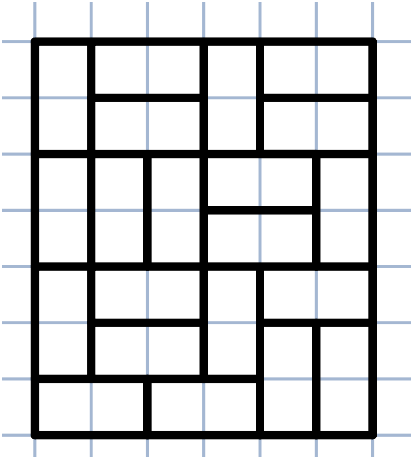
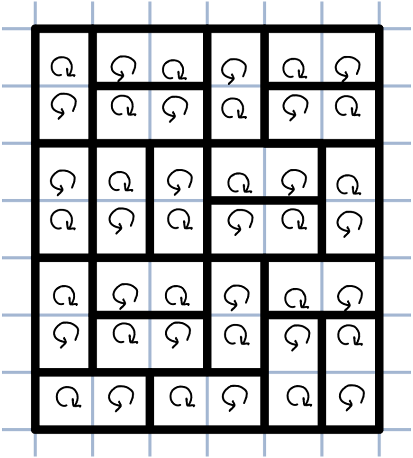
The starting point here is the height function introduced by Thurston (in 1990 following Conway&Lagarias's tiling groups).

Thurston's height functions



Thurston's height functions

Put a clockwise spiral on even sites and an anticlocwise spiral on odd sites.



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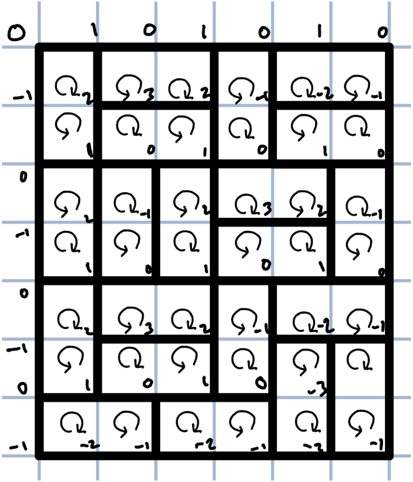
Now walk along the tiling increasing the height by 1 in the direction of the spiral.



While this seems extremely ad-hoc, underlying these height functions is some beautiful combinatorial group theory coming from Conway and Lagarias (which we won't have time for).

Surfaces associated with domino tilings

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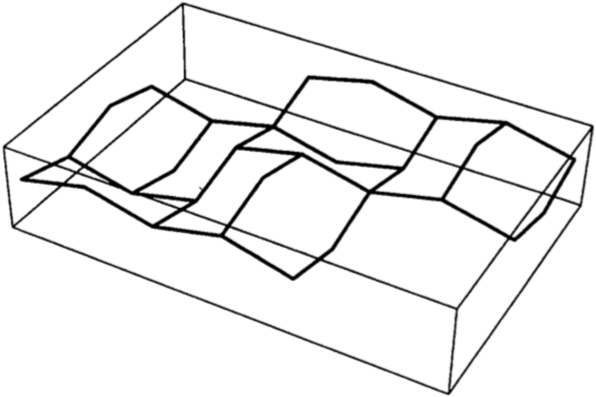

Figure : From Thurston's paper (1990)

Variational (**V**) and large deviations (**LD**) principle by Cohn, Kenyon and Propp, 2000

Theorem

Let $R^* \subset \mathbb{R}^2$ be bounded by a piecewise smooth simple closed curve. Fix a 2-Lipschitz height function h_b on ∂R^*. There exists an extension of h_b to a 2-Lipschitz function h_{max} on R^* such that the following holds.

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In this sense the function h_{max} above is the entropy maximiser with boundary conditions h_b.

Along the way, they also prove many properties of this entropy function like its strict convexity and continuity.

The effect of boundary conditions is, however, not entirely trivial and will be discussed in more detail in a subsequent paper.
(Kastelyn-1960)

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Dimer tilings of $\mathbb{Z}^2$

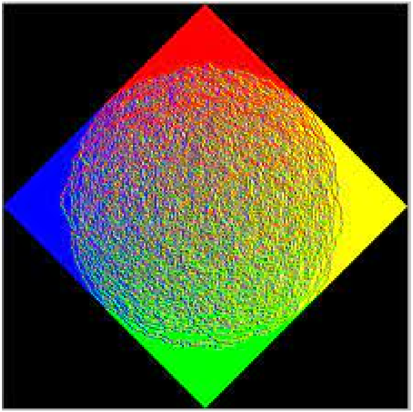

Figure : This was generated by Fusy and illustrates "the Artic circle phenomena"

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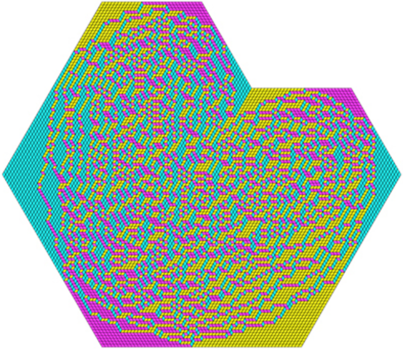


Figure : This was generated by Rick Kenyon and illustrates “the Artic circle phenomena”

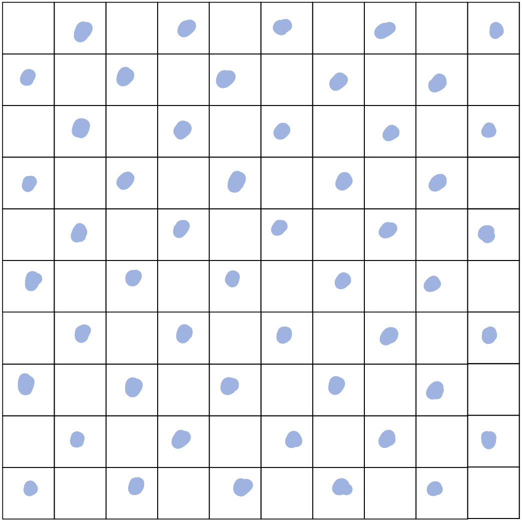
So instead of exact solvability we had to introduce softer techniques.

But what about the height functions? How can we even formulate the variational principle without them?

To this end, we define a discrete vector field associated with dimer tilings.

Discrete vector fields associated with dimer tilings

Label the even vertices of $\mathbb{Z}^3$ blue and the odd ones white.



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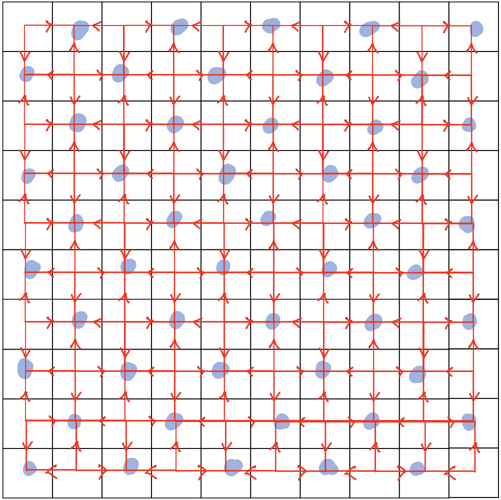
Now consider the flow growing from white to adjacent blue vertices of unit strength each.



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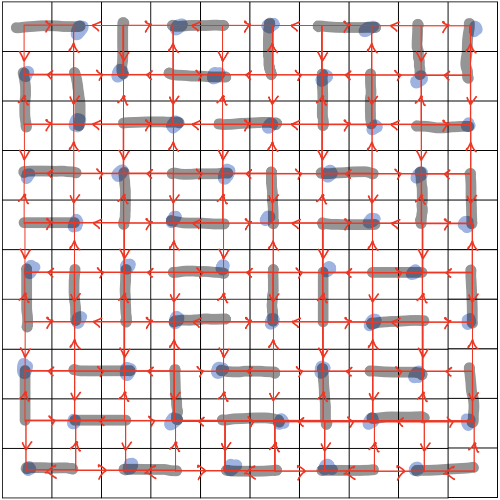
Now for a given a domino tiling keep the flow along those edges which are part of the tiling.



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These act like replacements of height functions but are far more difficult to work with.

Variational (**V**) and large deviations (**LD**) principle by C.,Sheffield and Wolfram

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*(**LD**) Further if g is any other divergence free extension of h_b taking values in $\mathcal{O}$ then the probability of t_n being close to g is exponentially small.*

The number of dimer tilings close to g is a function of g. The value of g captures "the average flux" of the Gibbs measure close to the dimer tiling at that point.

Variational (**V**) and large deviations (**LD**) principle by C.,Sheffield and Wolfram

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In this sense the function h_{max} above is the entropy maximiser with boundary conditions h_b.

We also need and prove various properties like strict convexity and continuity of the entropy as a function of the average flux.

One last complication: Gibbs measures with extremal slope

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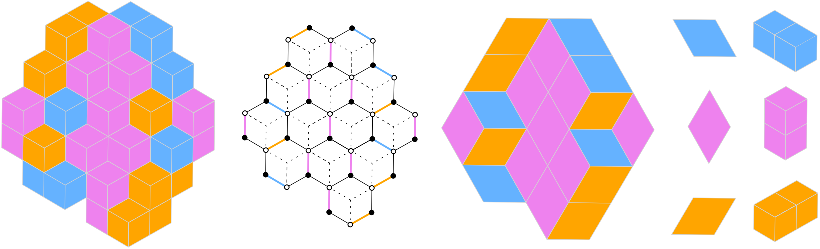
For $d = 3$, Gibbs measures of extremal "slope" decompose as lozenge tilings (which are important statistical physics models in their own right).

Lozenge tilings from extremal Gibbs measures on dimer tilings



Summary

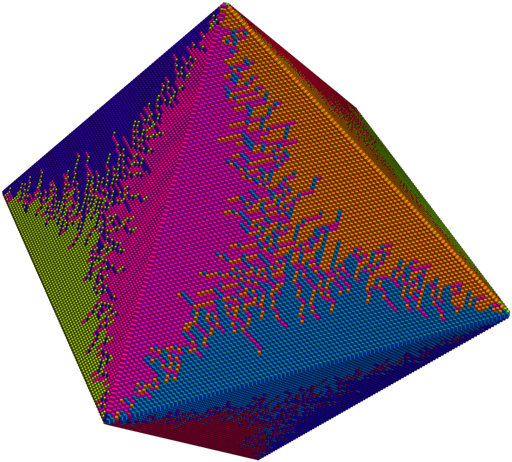
There are many things we now know about dimer tilings in three dimensions (and higher). For instance:

1. Ways to simulate uniform distribution on $\mathbb{Z}^3$.
2. The variational principle and the large deviations principle.
3. Nature of Gibbs measures with extremal “slope”.

And several things we don’t. For instance:

1. Exact solvability.
2. Whether any two tilings of a box can be connected by flips and trits.

Happy solving



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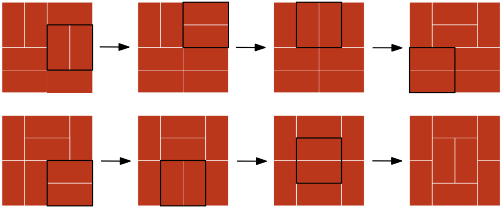
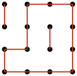
The important “suitable” hypothesis here is that one should be able to go from any given tiling of R_n to any other tiling using these local moves.

Flips: Local moves in two dimensions

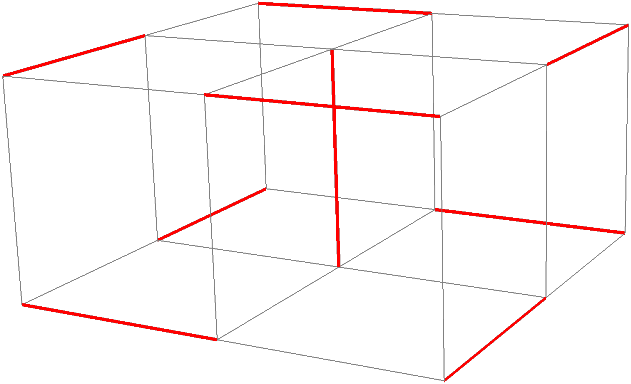
Given two adjacent dominos in the same direction we can always replace them by dominos in the perpendicular direction (but in the same plane). This is called a flip.





In two dimension any two domino tilings of a nice region R are connected by a series of flips.

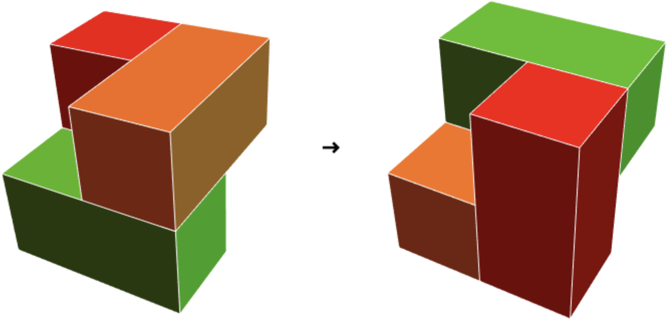
However in three dimensions, even tilings of boxes are not necessarily connected by a sequence of flips.



Clearly no flips are possible but there are many different possible tilings of this box. This was found by Freedman, Hastings, Nayak, and Qi in 2011.

Trits

It was realised however that by introducing another move called "trits", at least the tilings of these two layered boxes become connected to one another.



This was proved by Milet and Saldanha in 2017.

Flips and Trits

Question (Freire, Klivans, Milet and Saldanha, 2017)

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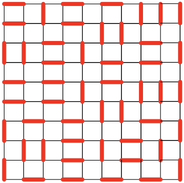
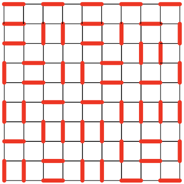
Yet we can construct credible simulations. The main idea for these simulations come from Broder (1986)- "How easy is it to marry at random?" with many similar variants going back all the way to Edmonds (1963)- "Paths, Flowers and Trees".

The main observation which helps us simulate: The double dimer model

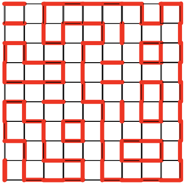
If we superimpose two dimer configurations on a finite region R then the edges either match up or they form loops of finite size.



TILING 1



TILING 2



The Superimposition.

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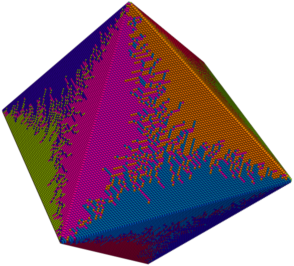

Double dimer model: Recent results

Recently Quitmann and Taggi (2022) proved that the double dimer model on higher dimensional torii $(d \geq 3)$ has macroscopic (long) loops.

This shows that the behaviour of the double dimer model in higher dimensions is very different from $d = 2$.