About Borel and almost Borel embeddings for $\mathbb{Z}^d$ actions

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February, Group Actions Seminar at UCSD

In this talk we will be reporting results with Tom Meyerovitch (2020) and ongoing work with Spencer Unger and also some with Scott Sheffield.

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By 'universal' we mean that 'any' free system (Y, S) (with low enough entropy) can be Borel embedded into (X, T).

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If you are not familiar with entropy think of it as “size” for the time being. We will come back to it later in the talk to provide more intuition.

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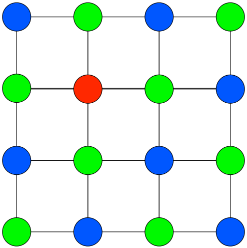

Figure : Shift action.

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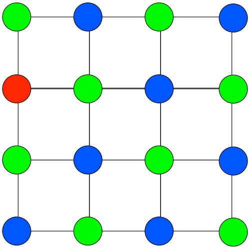

Figure : Moving to the left, $\sigma^{(1,0)}$.

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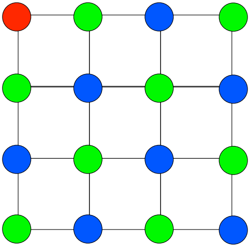

Figure : Moving up $\sigma^{(0,-1)}$.

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Theorem (Quas, Soo 2012)

Let $T : (\mathbb{R}/\mathbb{Z})^d \to (\mathbb{R}/\mathbb{Z})^d$ *be a ergodic toral automorphism. Then* $((\mathbb{R}/\mathbb{Z})^d, T)$ *is 'ergodic' universal.*

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Theorem (Robinson and Şahin, 2003)

If a shift space (X, σ) has strong enough 'mixing' conditions then it is ergodic universal.

The direction of our work

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This answered various questions raised by Robinson-Şahin (2003), Quas-Soo (2012), Gao-Jackson (2015) and Boyle-Buzzi (2017) and recovered a result by Burguet (2020) and by Gao-Jackson-Krohne-Seward.

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Entropy : A swift introduction

Partitions and their refinement

Let (X, T) be a $\mathbb{Z}^d$ dynamical system. Given a partition P of X, for boxes $[1, n]^d$ we will consider the refined partition

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2. $\mu(Q) > 1 - \epsilon$.

If X is compact, this Gurevich entropy is the same as the topological entropy.

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This is the starting point of a long list of questions which lie at the heart of going from almost universality to universality.

Entropy for shift spaces

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The space (X, T) has (Gurevich) entropy $h+ : h(X, T)$ if it is the smallest number for which

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If X is a shift space then it can be calculated by the following simple (but often difficult to compute) formulae.

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So the entropy of $(A^{\mathbb{Z}^d}, \sigma)$ is $\log |A|$.

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Some examples of shift spaces.

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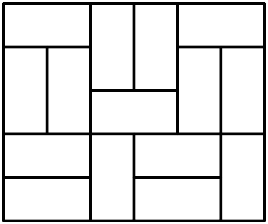

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For example tilings by boxes exactly one of whose side length is $k+1$ and the rest are k gives us a coprime box shift.

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If (Y, S) is a shift space whose entropy is lower than that of the domino tilings then (up to periodic points) there is a Borel embedding from (Y, S) to the space of domino tilings.

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What does one need for universality?

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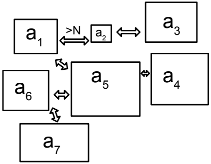

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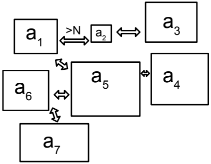

This was disappointing because nothing like this can hold for domino tilings.

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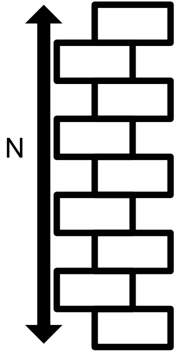
This pattern completely determines what can be placed for distance N/2



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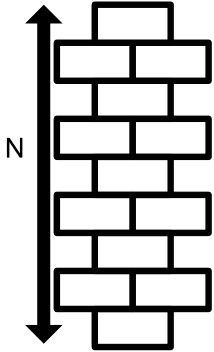

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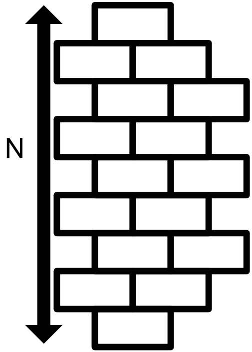

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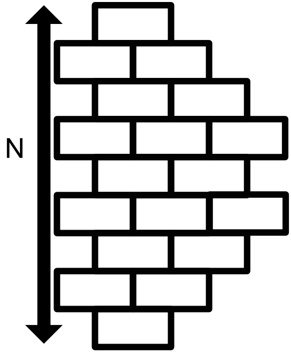

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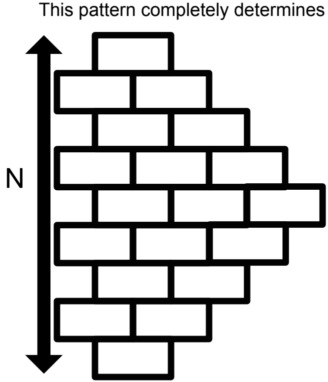

But Benjy is always right.

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Given a shift space X and shape $B \subset \mathbb{Z}^d$, we denote by $\mathcal{L}(X, B)$ the set of patterns appearing on the shape B, that is,

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$$\mathcal{L}(X, B) := \{x|_B \ : \ x \in X\}.$$

Flexible sequence

Given a shift space X a flexible sequence with gap k and scaling N is a sequence

$$\mathcal{C} = (C(N.B) \subset \mathcal{L}(X, N.B); B \text{ is a box})$$

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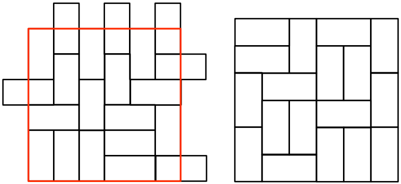
For domino tilings the gap is $6d$, the scaling is 2 and $C(2.B)$ is the set of proper domino tilings of the box $2.B$.

For coprime box tilings the scaling is the product of lengths of the sides of the boxes (say N), the gap is $6Nd$ and $C(N.B)$ is the set of proper tilings of the box $N.B$.

Flexible sequence for domino tilings



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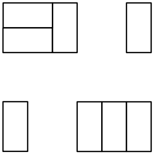

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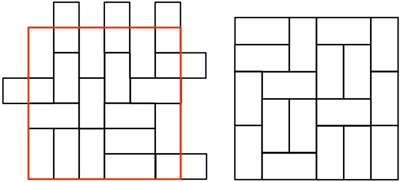

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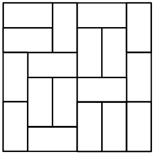

Flexible sequence for other spaces

Given a shift space X a flexible sequence with gap k and scaling N is a sequence

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such that for all boxes $B, B_1, \ldots, B_t$ for which

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A large class of spaces have flexible sequences: Proper 3-colourings, space of graph homomorphisms, space of self-avoiding walks on the $\mathbb{Z}^d$ lattice, directed bi-infinite Hamiltonian paths and so on.

While we defined this for shift spaces, a corresponding (and much more involved) notion exists for general topological dynamical systems.

But what is flexibility good for?

Given a flexible sequence

$$\mathcal{C} = (C(N.B); B \text{ is a box})$$

of scaling N, its entropy is given by

$$h(\mathcal{C}) := \lim_{n\to\infty} \frac{1}{N^d n^d} \log |C(N.[1, n^d])|.$$

Theorem (Chandgotia, Meyerovitch 2020)

If X is a shift space with a flexible sequence of entropy $h(\mathcal{C})$ then for any free dynamical system (Y, S) of entropy less than $h(\mathcal{C})$ there exists an embedding from (Y, S) to (X, σ) up to a universally null set.

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For a particular system one may ask: Is $h(\mathcal{C}) = h(X, T)$? This is a key question and of extremely challenging nature.

Is $h(\mathcal{C}) = h(X, T)$? Domino tilings

It follows from work by Kastelyn (1961), Temperley-Fisher (1961) and Burton-Pemantle (1993) that

$$\frac{1}{(2N)^2} \log \left(|\text{domino tilings of a } [1, 2N]^2| \right) \to h(\text{domino tilings}, \sigma),$$

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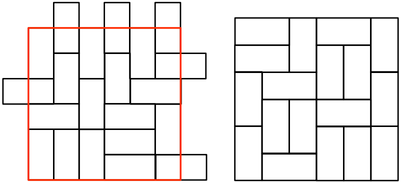

in the computation of entropy we only need to care about the patterns as on the right.

In ongoing work with Scott Sheffield, we have extended this result to higher dimensions.

We know close to nothing about $h(\mathcal{C})$ for general coprime shifts.

Question

Let $\mathbb{T}$ be a set of coprime boxes. Let N be the product of length of the sides of $\mathbb{T}$. Prove that

$$\lim_{n\to\infty} \frac{1}{N^d n^d} \log(\textit{the number of tilings of } [1, Nn]^d \textit{ by elements of } \mathbb{T}) = \textit{ topological entropy of all the tilings by } \mathbb{T}.$$

This gives you a rough idea of the combinatorial challenges involved in proving this result.

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Now let us talk about how we can push the results from the almost Borel world to the Borel world. For this we needed a result by Gao-Jackson-Krohne-Seward.

An analogue of Rokhlin's lemma by Gao, Jackson, Krohne and Seward

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This was the key to pushing our result to the Borel category.

For this we needed a stronger notion of flexibility.

Strong flexibility

Given a shift space X a flexible sequence

$$\mathcal{C} = (C(N.B); B \text{ is a box})$$

with gap k and scaling N is a sequence of

a special set of patterns $C(N.B)$ which appear in X for a box $N.B$

such that for all boxes $B, B_1, \ldots, B_t$ for which

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If (X, σ) has a strongly flexible sequence $\mathcal{C}$ and (Y, S) is any free $\mathbb{Z}^d$ dynamical system then there is a factor from (Y, S) to the free part of (X, σ). There is no need to get rid of the universally null set.

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If (Y, σ) is a shift space whose entropy is lower than that of $h(\mathcal{C})$ then there is a Borel embedding from (Y, σ) into (X, σ).

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1. Entropy : A swift introduction ✓
2. Some shift-invariant spaces : Concentrating on the ones I love ✓
3. What do we need for universality: Some hard combinatorial question which are simple to state!✓
4. Some open questions: Why we have barely gotten started!

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