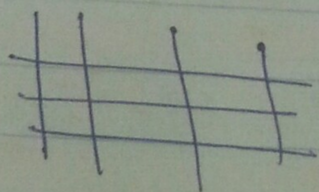
BGA. - talk.

A - finite set (discrete topology)

$\mathbb{Z}^d$ - Cayley graph of $\mathbb{Z}^d$ with standard generators. $\boxed{d \geq 2}$ for the talk!

$\mathbb{Z}^2 =$ 

__Pattern__ an element of A^A (for some $A \subset \mathbb{Z}^d$ finite)

$\mathcal{F}$ be a set of patterns

$X_{\mathcal{F}} := \{x \in A^{\mathbb{Z}^d} \mid$ translates of patterns in $\mathcal{F}$ do not occur in $x\}$.

$\hookrightarrow$ Shift spaces.

If $\mathcal{F}$ can be chosen finite $X_{\mathcal{F}}$ is called ~~SFT~~ (a shift of finite type (SFT)

~~The non-emptiness problem for SFT's is undecidable.~~

(say) The non-emptiness problem for SFT's is undecidable. $\therefore$ So we study a more restricted class.

<u>Hom - shift</u>: H - finite undirected graph without self-loops. (Connected always)

G - maybe infinite.

~~A graph~~ $\sim_{G/H}$ $\rightarrow$ adjacency in G/H (say, we drop subscript when $G = \mathbb{Z}^d$)

$f : G \longrightarrow H$ homomorphism if $i \sim_G j \Rightarrow f(i) \sim_H f(j)$.

$Hom(G, H) :=$ space of all homomorphism from G to H.

K_n - complete graph with vertices $1, - n$

$Hom(G, K_n)$ - ~~complete~~ proper n-colourings of G

$Hom(G, \circ\!-\!1)$ - hard core model (no two 1s are adjacent)

$X_H \doteq \mathrm{Hom}(\mathbb{Z}^d, H)$ ..

— shift space is called a hom-shift

(say) why is this a shift space?

The language $\mathcal{L}(X)$ is the set of all patterns appearing in X.

~~B_n is a box in $\mathbb{Z}^d$ with sidelength n~~

$\mathcal{L}_A(X) \doteq \mathcal{L}(X) \cap A^A$; B_n box of sidelength n.

topological entropy

$$h_{top}(X) \doteq \lim_{n \to \infty} \frac{\log |\mathcal{L}_{B_n}(X)|}{|B_n|}.$$

Entropy minimality $Y \subseteq X$

then $h_{top}(Y) \leq h_{top}(X)$

X - entropy minimal if

$Y \subsetneq X \Rightarrow h_{top}(Y) < h_{top}(X)$

(say)

That is, if we forbid a pattern entropy drops.

Qn: When is X_H entropy minimal?

Result: C_n - n-cycle with vertices $0, \ldots, n-1$

(C., Meyerowitch) '13

- X_{C_n} is entropy minimal.

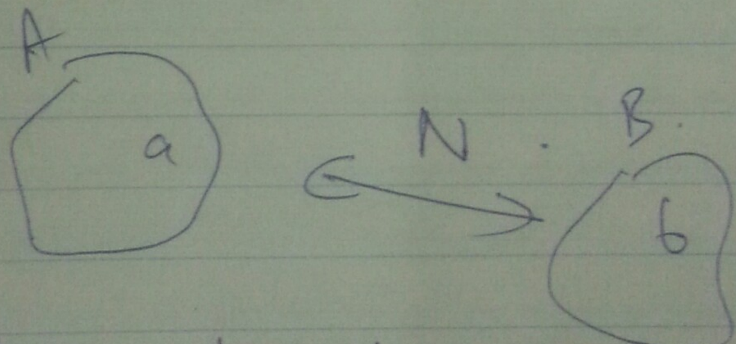
(C. '14) H - four-cycle free if C_4 is not a subgraph of H.

H - four-cycle free ~~H four~~ $\Rightarrow X_H$ is entropy minimal.

Previous results:

~~Transitivity~~ ~~S.I.~~

X is strongly irreducible (S.I.) $\exists N$ st $\forall$ $a \in L_A(X)$, $b \in L_B(X)$ ~~$a, b \in L(X)$~~ st

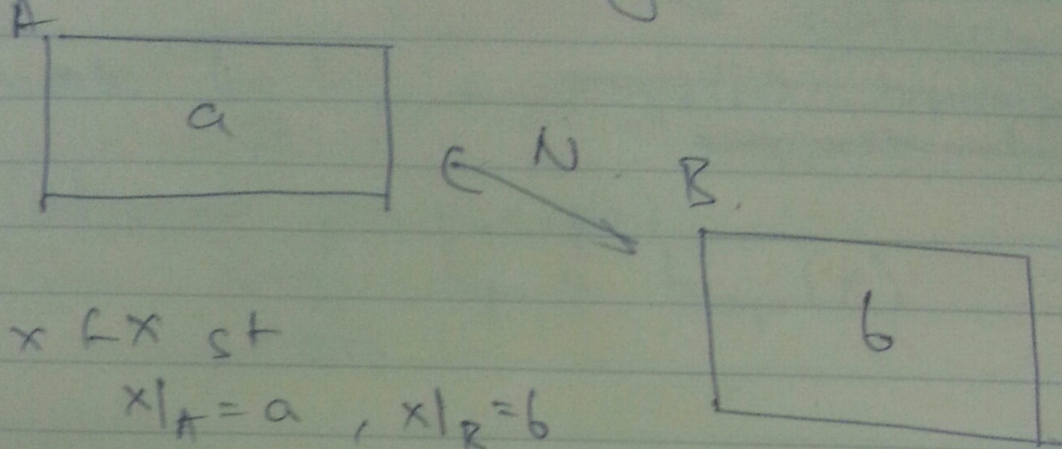


$\exists x \in X$ st $x|_A = a$, $x|_B = b$.

(Schraudner '09) S.I shift space is entropy minimal. (say stronger resu

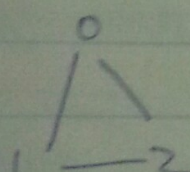
Qn When is X_H S.I?

Block-gluing X is block-gluing if $\exists N$ st. $a \in \mathcal{L}_A(X)$, $b \in \mathcal{L}_B(X)$ ~~$a, b \in \mathcal{L}(X)$~~ rectangular st



$\exists\, x \in X$ st
$x|_A = a$, $x|_B = b$

Boyle, Pavlov, Schraudner '09 Block-gluing $\not\Rightarrow$ entropy minimal.

C_3 — 

X_{C_3} is not block-gluing (say) yet it is entropy minimal.

μ - shift - invariant probability measure.

Can associate entropy measure theoretic entropy

denote by h_μ. ~~supp(μ) - topological support of μ~~.

(Variational principle)

$$\sup_{\mu(X)=1} h_\mu = h_{top}(X)$$

$\exists\ \mu$ which achieves this max. Called (MME)

For such μ

$$h_\mu = h_{top}(X).$$

supp (μ) = smallest closed set Y st $\mu(Y) = 1$.

~~μ is an MME~~. if and only if.

Observe: ~~μ is a~~ X is entropy minimal iff

for all mme. μ of X; supp$(\mu) = X$.

Can prove. further: X. SFT. μ m.m.e $\Rightarrow \mu$ adapted

that is, ~~μ. mme. of X~~, $x \in$ supp(μ)

~~$y, x \in X$~~ $y \in X$ differs at finitely many sites from x

$$\Rightarrow y \in \text{supp}(\mu).$$

(Give examples).

$\rightarrow$ μ nice

Want to prove $h_\mu > 0$. μ adapted to ~~some~~ of X_{C_3} ~~(nice adapted measure)~~ $\rightarrow$ $\text{supp}(\mu) = X_{C_3}$.

(7)

~~For~~

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[d, "\text{mod } 3 \text{ covering map}"] \\
\mathbb{Z}^2 \arrow[ur] \arrow[r, "x"'] & C_3
\end{tikzcd}$$

$\forall f \in X_{C_3} \exists \tilde{x} \in X_{\mathbb{Z}}$ s.t.

$\tilde{x} \bmod 3 = x$.

$\tilde{x}$ is unique given $\tilde{x}(\vec{0})$.

In fact, μ- ~~some~~ ergodic,

we can associate slopes for every direction $\vec{e_1} \ldots \vec{e_n}$

$$sl_{\vec{e}_1}(x) \doteq \frac{\tilde{x}(n\vec{e}_1) - \tilde{x}(\vec{0})}{n}$$

exists and is constant μ-a.e.

~~Repla~~

~~$st_{\vec{e_i}}(\mu)$~~

If μ such that $|sl_{\vec{e_i}}(x)| = 1$ a.e. for some

Then $\tilde{x}_{\vec{0}}$ and $\tilde{x}_{n\vec{e_i}}$

completely determine

$\tilde{x}_{\vec{e_i}}, \tilde{x}_{2\vec{e_i}} \ldots \tilde{x}_{(n-1)\vec{e_i}}$. $h_\mu = 0$

On the other hand!

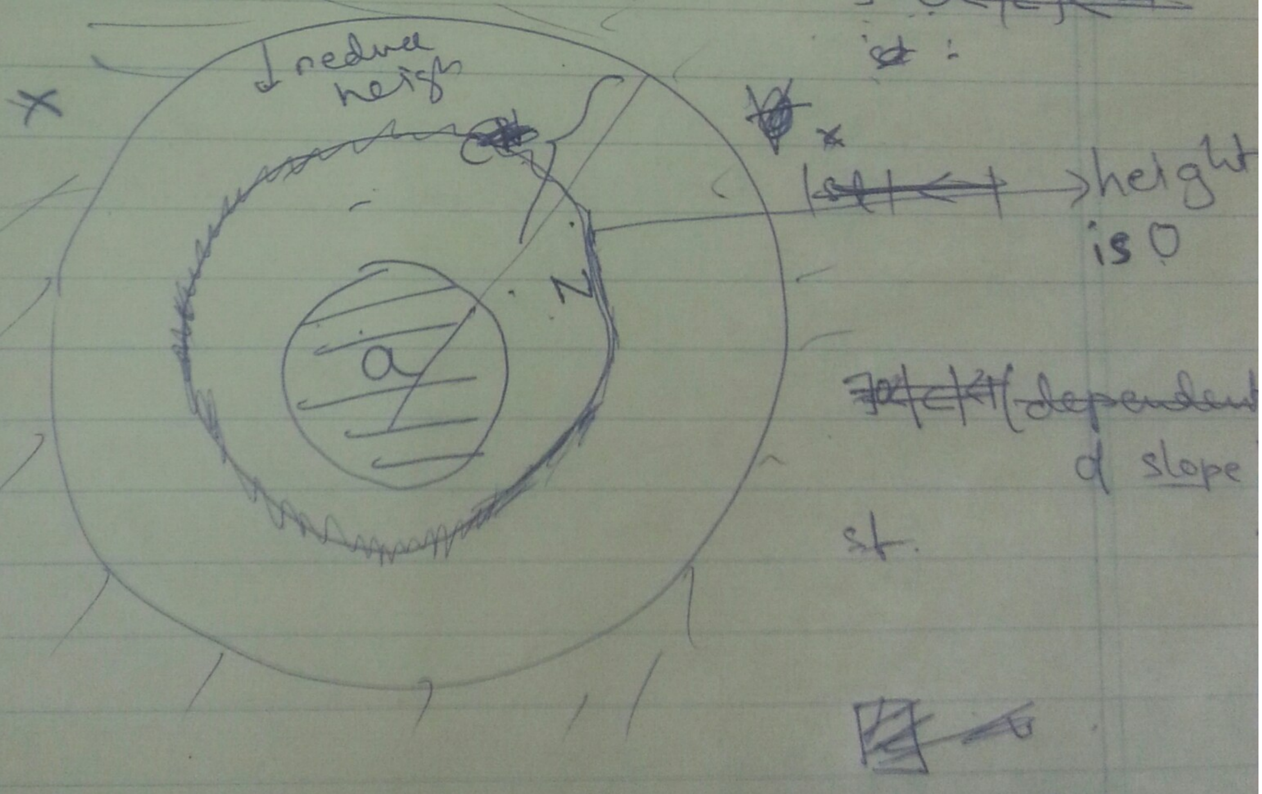
If μ such that $|sl_{\vec{e_i}}(x)| < 1 \; \forall i$ $\Rightarrow \overline{\mu \text{ not nice}}$

$\Rightarrow$ μ adapted $\Rightarrow$ $supp(\mu) = X_{\ell_3}$.

underlying mixing condition.

~~$\exists\, 0 < |c| < 1$~~

~~st~~:



~~$\exists\, |c| < 1$ (dependent~~ of slope

st.

$\forall x, |sl_{\vec{e_i}}(x)| < 1 \; \forall i$, $a \in \mathcal{L}_A(x)$ $\exists N, y \in X$

st. $y|_A = a$ $\quad y|_{B_N^c} = x|_{B_N^c}$.

$\Rightarrow y \in supp(\mu) \Rightarrow \mu([a]) > 0 \Rightarrow \mu\, supp(\mu) = X_{\ell_3}$

$$\begin{tikzcd} & \tilde{H} \arrow[d, "\text{projection } \pi"] \\ \mathbb{Z}^2 \arrow[ur, "\tilde{x}"] \arrow[r, "x"'] & H \end{tikzcd}$$

$\tilde{H}$ – universal cover

$H \to$ four-cycle free

Can find lifts. . . .

<u>Conjecture:</u> For all H.

$\mathrm{Hom}(\mathbb{Z}^d, H)$ is entropy minimal!