Maryland Penn State Conference

$d \geq 2$.

$\mathcal{A}$ - finite alphabet.

Shape - finite $F \subset \mathbb{Z}^d$.

Pattern - element of $\mathcal{A}^F$ for some shape F.

Configuration - elements of $\mathcal{A}^{\mathbb{Z}^d}$

Notation: $x \in \mathcal{A}^{\mathbb{Z}^d}$, $\vec{j}, \vec{i} \in \mathbb{Z}^d$.

$x_{\vec{i}} \doteq x(\vec{i})$, $\sigma^{\vec{j}} : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$.

$(\sigma^{\vec{j}}(x))_{\vec{i}} \doteq x_{\vec{i}+\vec{j}}$.

Forbidden list. - $\mathcal{F}$ a set of patterns. (always finite).

$X^d_{\mathcal{F}} \doteq \{ x \in \mathcal{A}^{\mathbb{Z}^d}$: patterns from $\mathcal{F}$ do not appear in x or translates of x. $\}$.

shift of finite type. (SFT).

$\mathbb{Z}^d$ - also mean Cayley graph with respect to standard generators $\vec{e}_1, \vec{e}_2 \ldots \vec{e}_d$.

By a recoding argument, we can assume $\mathcal{F}$ - set of patterns on edges and vertices of $\mathbb{Z}^d$. (for a different alphabet A_1).

In this case, $X^d_{\mathcal{F}}$ nearest neighbour SFT.

Note: $X^d_{\mathcal{F}}$ is invariant under shifts $\{\sigma^{\vec{v}} : \vec{v} \in \mathbb{Z}^d\}$.

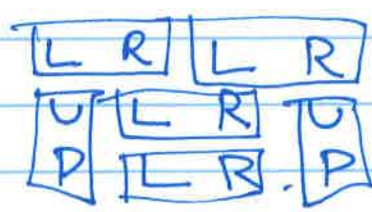
Example. $d = 2$.

$$A_1 = \{U, D, L, R\}$$

$$\mathcal{F} = \left\{ \begin{smallmatrix} v \\ D \end{smallmatrix} : v \neq U \right\}$$

$$\cup \left\{ \begin{smallmatrix} U \\ v \end{smallmatrix} : v \neq D \right\}$$

$$\cup \{ L\,v : v \neq R \} \cup \{ v\,R : v \neq L \}.$$



$X^2_{\mathcal{F}}$ - Domino tilings of $\mathbb{Z}^2$.

Hom-shift

$\mathcal{H}$ - finite undirected graph. (without multiple edges, isolated vertices).

Eg.

$$Hom(\mathbb{Z}^d, \mathcal{H}) := \{ x : \mathbb{Z}^d \to \mathcal{H} \mid \vec{i} \sim_{\mathbb{Z}^d} \vec{j} \Rightarrow x_{\vec{i}} \sim_{\mathcal{H}} x_{\vec{j}} \}.$$

adjacency relation

hom-shift.

- nearest neighbour SFT. where

$$\mathcal{F}_{\mathcal{H}} = \{ a \in \mathcal{H}^{\{\vec{0}, \vec{e_i}\}} : a_{\vec{0}} \not\sim_{\mathcal{H}} a_{\vec{e_i}} \}$$

for $i = 1 \ldots d$

$X^d_{\mathcal{H}} = Hom(\mathbb{Z}^d, \mathcal{H})$

(by looking at 1×2, 2×1 periodic points, domino tilings is not conjugate to a hom-shift)

Examples:

* Hard-square model.

Hom$(\mathbb{Z}^d, \{0 - 1\})$.

— no two 1's can be adjacent.

* ~~Par.~~ n-coloured chessboard.

Hom$(\mathbb{Z}^d$, complete graph with n vertices$\}$.

n=3

— adjacent symbols are distinct.

n=4

Say (precisely the nearest neighbour SFT with the same constraints in every direction).

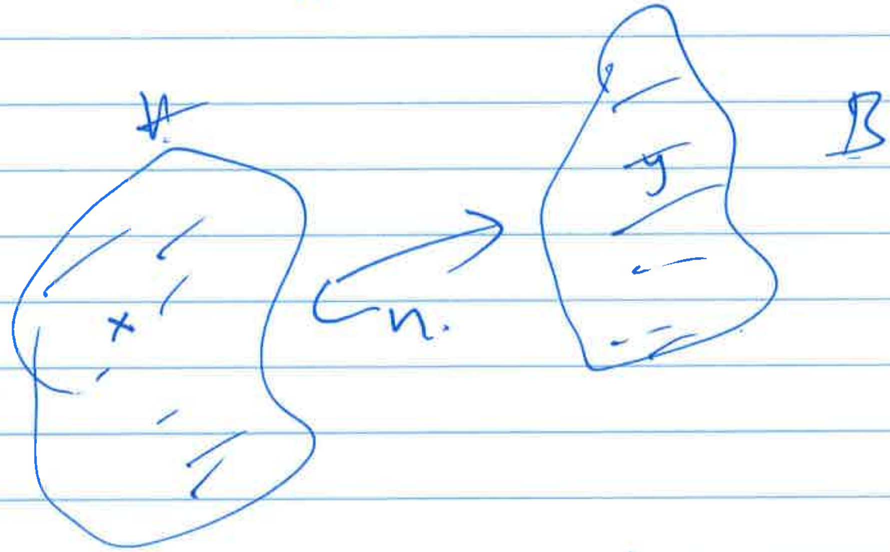
$d \geq 2$

	Nearest Neighbour SFTs. $X^d_{\mathcal{F}}$	Hom-shifts. $X^d_{\mathcal{H}}$. $Hom(\mathbb{Z}^d, \mathcal{H})$
Non-emptiness	Undecidable.	iff $\mathcal{H}$ has an edge or a self-loop.
periodic points.	~~It is possible X^d~~ might be empty for even $X^d_{\mathcal{F}}$ with positive entropy.	$P_n(X^d_{\mathcal{H}}) := \{x \in X \mid \sigma^{n\vec{e}_i}(x) = x \text{ for } i = 1, \ldots d\}$ $\frac{1}{\lvert P_n(X^d_{\mathcal{H}}) \rvert} \sum_{x \in P_n(X^d_{\mathcal{H}})} \delta_x$ accumulates to measures of maximal entropy (Friedland 1997)
Entropy:	Right recursively enumerable numbers (Hochman & Meyerovitch) 2010 (say what these are)	Computable (Friedland 1997)
Measures of maximal entropy (mme)	~~Can~~ ~~It~~ Can be uncountably many ergodic mmes.	Conjecture: $\exists$ atmost finitely many ergodic mmes
Mixing.	Undecidable.	$\mathcal{H}$ not bipartite & connected: Mixing $\mathcal{H}$ bipartite, & connected: $X = X^1 \cup X^2$, X^i mixing wrt $(2\mathbb{Z})^d$ action.

<u>What could go wrong?</u>

X is strongly irreducible (SI) if $\exists n$ s.t. for all ~~shapes~~ shapes A, B separated by distance n. $\forall$, $x, y \in X$, $\exists z \in X$ st

$$z|_A = x|_A \text{ and } z|_B = y|_B$$



Qn: When is $\mathrm{Hom}(\mathbb{Z}^d, H)$ strongly irreducible?

If H is bipartite then

$\mathrm{Hom}(\mathbb{Z}^d, H)$ is not SI.

So we introduce phased SI where x can be replaced by $\sigma^{\vec{e_i}}(x)$ if necessary.

$$H = \begin{tikzcd}[column sep=small] & 0 \arrow[dl, dash] \arrow[dr, dash] & \\ 1 \arrow[rr, dash] & & 2 \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, "\bmod 3"] \\ \mathbb{Z}^d \arrow[ur, "\exists \tilde{x}"] \arrow[r, "\forall x"'] & \begin{smallmatrix} & 0 & \\ 1 & & 2 \end{smallmatrix} \end{tikzcd}$$

~~finding such a lift implies~~
This covering implies Hom $(\mathbb{Z}^d, \begin{smallmatrix} & 0 & \\ 1 & - & 2 \end{smallmatrix})$ is not phased SI.

Thm (Ch., Marcus) H has no four-cycles and self-loops

Hom$(\mathbb{Z}^d, H)$ is phased SI iff H is a tree.

Given any graph H. $\exists$ ~~graph~~ maximum

Covering graph H: H_{max} st

$$\begin{tikzcd} & H_{max} \arrow[d, "\text{covering map}"] \\ \mathbb{Z}^d \arrow[ur, "\exists \tilde{x}"] \arrow[r, "\forall x"'] & H \end{tikzcd}$$

H_{max} - unique, normal cover.
(upto isomorphism)

H - has no four-cycles & self-loops H_{max} - universal cover of H.

If $|H_{max}| = \infty$ then $Hom(\mathbb{Z}^d, H)$ is not S1.

<u>Thm</u>: (c.) It is undecidable whether

$|H_{max}| = \infty$, $H_{max} = H$ ...

<u>Conjecture</u>: It is undecidable whether

$Hom(\mathbb{Z}^d, H)$ is phased S1/S1.