Markov Random Fields and Gibbs States

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Outline

- Homomorphism spaces
- Markov random fields and Gibbs states
- When are Markov random fields Gibbs states?
- Describing conditions on the support
  - All Markov random fields are Gibbs: Dismantlable graphs and the 3-coloured chessboard
  - Not all Markov random fields are Gibbs: The square island shift
- The pivot property

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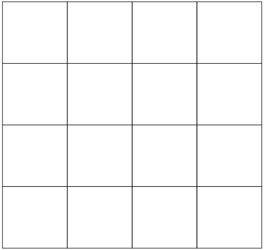

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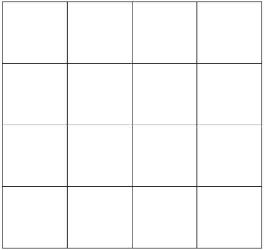

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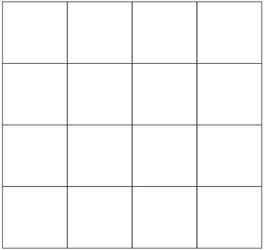

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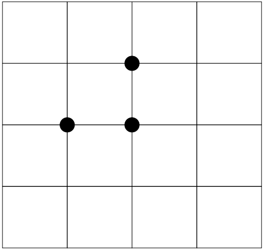

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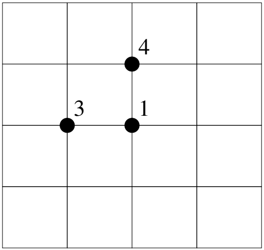

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Cylinder set $[4,3,1]_A$

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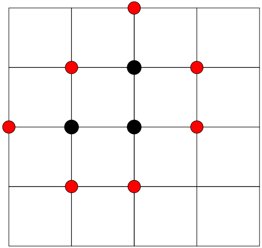
$$\begin{aligned} [a]_A &:= \{x \in X \mid x|_A = a\} \text{ (Cylinder set)} \\ \partial A &:= \{v \in \mathcal{V}_{\mathcal{G}} \setminus A \mid v \sim w \in A\} \text{ (Boundary).} \end{aligned}$$



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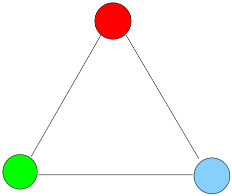

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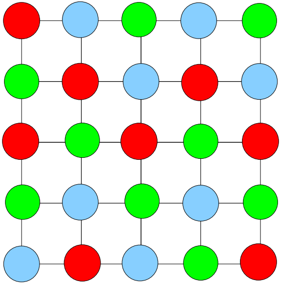

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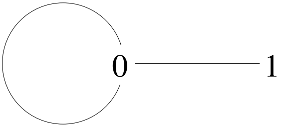

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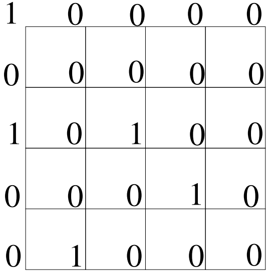

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X has a safe symbol $\star$ if for all $x \in X$ and $n \in \mathcal{V}_{\mathcal{G}}$, the configuration y given by

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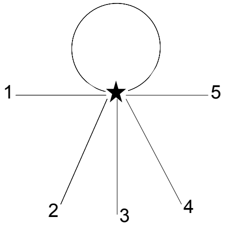

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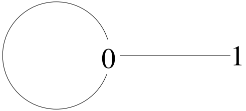

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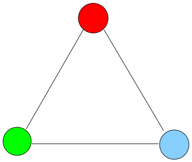

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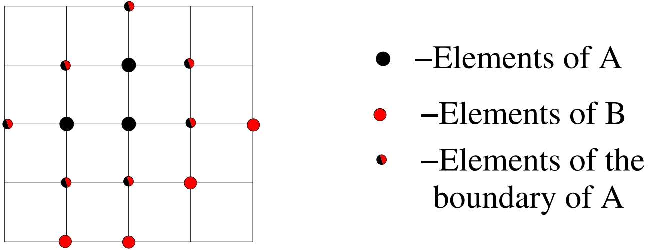

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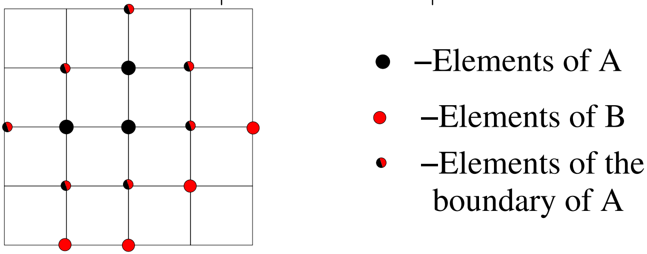

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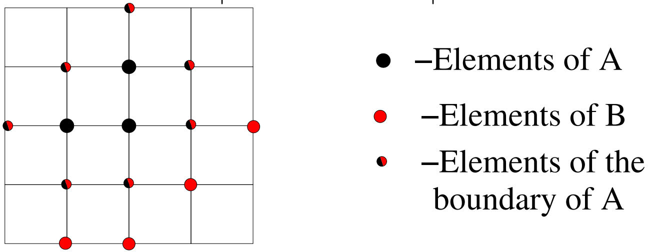

The set of conditional measures $\mu([\cdot]_A \mid [b]_{\partial A})$ for all $A \subset \mathcal{V}_{\mathcal{G}}$ finite and $b \in \mathfrak{A}^{\partial A}$ is called the specification for the measure μ. It might not have any finite description.

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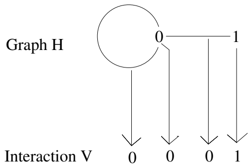

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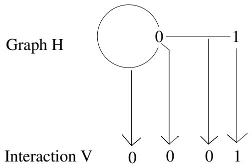

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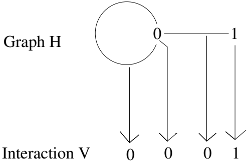

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$$\mu([x]_A \,\Big|\, [x]_{\partial A}) = \frac{\prod\limits_{C \subset A \cup \partial A} e^{V([x]_C)}}{Z_{A,x|_{\partial A}}} = \frac{e^{\text{number of } 1's \text{ in } x|_{A\cup\partial A}}}{Z_{A,x|_{\partial A}}}.$$

Question: Under what conditions on the support is every MRF a Gibbs state for some n.n. interaction?

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New Results:

- The support is the 3-coloured chessboard model.
- A generalisation of the Hammersley-Clifford theorem when $\mathcal{G}$ is bipartite.

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New Results:

- For $\mathcal{G} = \mathbb{Z}^2$ we constructed a family of shift-invariant MRFs which are not Gibbs for any shift-invariant finite-range interaction (not just nearest neighbour).

Dismantlable Graphs

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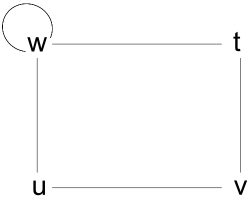

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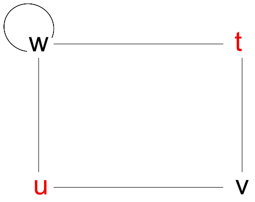

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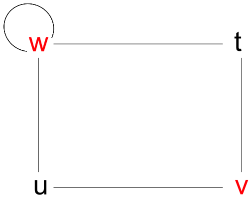

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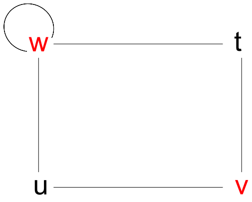

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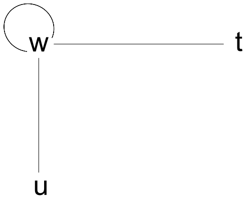

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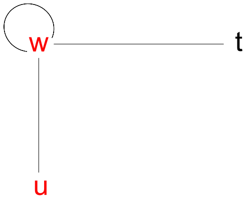

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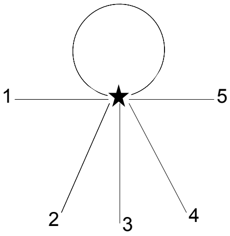

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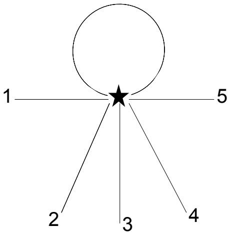

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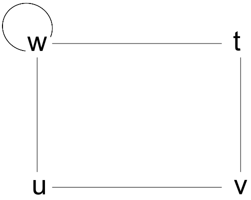
However there are dismantlable graphs $\mathcal{H}$ even if $Hom(\mathbb{Z}^d, \mathcal{H})$ does not have a safe symbol.



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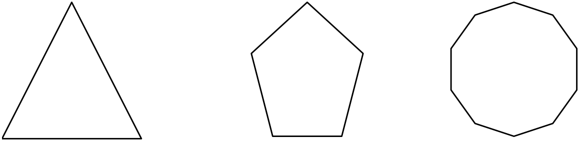
And there are graphs where no folding is possible. Let C_n denote the n-cycle.



C_3 $\quad$ C_5 $\quad$ C_{10}

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How can such a theorem be proved?

Pivot Property

A space X is said to satisfy the pivot property if for all $x, y \in X$ which differ only on finitely many sites there exists a chain $x = x^1, x^2, x^3, \ldots, x^n = y \in X$ such that x^i, x^{i+1} differ on at most a single site.

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A space X is said to satisfy the generalised pivot property if there exists $K > 0$ such that for all $x, y \in X$ which differ only on finitely many sites there exists a chain $x = x^1, x^2, x^3, \dots, x^n = y \in X$ such that x^i, x^{i+1} differ only on a region of diameter at most K.

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- Domino tilings.

The 3-coloured Chessboard

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The 3-coloured chessboard has the pivot property.

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0	2	0	1	2	1	0
1	0	1	0	1	0	1
0	1	0	2	0	1	2
2	0	1	0	1	2	0
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Thus the space of specifications on $supp(\mu)$ can be parametrised by finitely many parameters.

Question: Suppose we are given a nearest neighbour shift of finite type with the pivot property. Is there an algorithm to determine the number of parameters which describes the specification?

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$v_3 = \frac{\mu(\begin{bmatrix} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{bmatrix})}{\mu(\begin{bmatrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{bmatrix})}$.

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$$
\begin{aligned}
v_1 &= exp(V(01) + V(10) + V(\tbinom{0}{1}) + V(\tbinom{0}{1}) \\
&\quad - V(21) - V(12) - V(\tbinom{2}{1}) - V(\tbinom{1}{2})), \\
v_2 &= exp(V(12) + V(21) + V(\tbinom{2}{1}) + V(\tbinom{1}{2}) \\
&\quad - V(02) - V(20) - V(\tbinom{0}{2}) - V(\tbinom{2}{0})), \\
v_3 &= exp(V(02) + V(20) + V(\tbinom{2}{0}) + V(\tbinom{0}{2}) \\
&\quad - V(01) - V(10) - V(\tbinom{0}{1}) - V(\tbinom{1}{0})).
\end{aligned}
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μ is Gibbs if and only if $v_1 v_2 v_3 = 1$.

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What if the pivot property does not hold?

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Thus the Hammersley-Clifford type conclusion holds for fully supported measures.

What if the pivot property does not hold? Every 1 dimensional nearest neighbour shift of finite type has the generalised pivot property.

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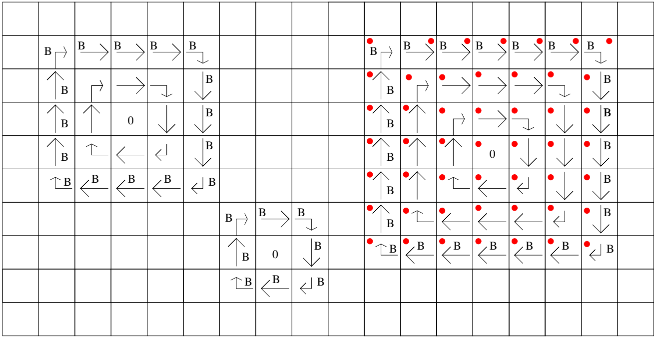
.

There are two kinds of squares: ones with red dots and ones without red dots which float in a sea of blanks.

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Theorem (Chandgotia and Meyerovitch '13)

There exists a shift-invariant MRF supported on the square island shift which is not Gibbs for any shift-invariant finite-range interaction.

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Is there a more natural example?

Theorem (Tim Austin '15)

There is a graph $\mathcal{H}$ for which Hom$(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property.

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Question: Is there a shift-invariant Markov random field which is supported on $Hom(\mathbb{Z}^2, \mathcal{H})$ which is not Gibbs for some shift-invariant finite range interaction?

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Question: Is there a shift-invariant Markov random field which is supported on *Hom*$(\mathbb{Z}^2, \mathcal{H})$ which is not Gibbs for some shift-invariant finite range interaction?

Question: Can you classify graphs $\mathcal{H}$ for which *Hom*$(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property?

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Question: Can you classify graphs $\mathcal{H}$ for which $Hom(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property?

Question: If $Hom(\mathbb{Z}^2, \mathcal{H})$ has the generalised pivot property, can you determine the minimum number of parameters required to determine the specification of a Markov random field?

Thank You!