The Pivot Property for $Hom(\mathbb{Z}^d, \mathcal{H})$

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Outline

- Homomorphism Spaces
- Pivot Property
- Dismantlable Graphs
- Complete Graphs
- Four-cycle Free Graphs and the Universal Cover
- Generalised Pivot Property

Homomorphism Spaces

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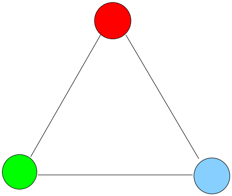
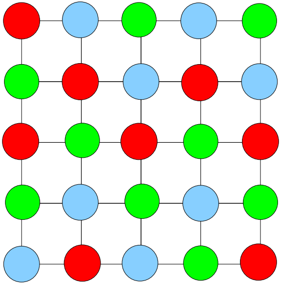


Graph $\mathcal{H}$



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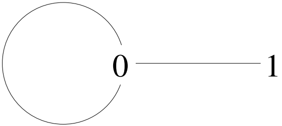

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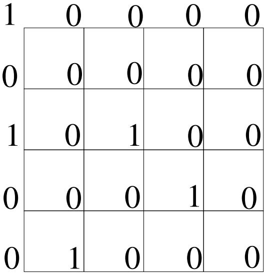

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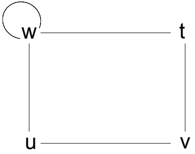
- $Hom(\mathbb{Z}^d, \mathcal{H})$ is said to have the pivot property if for all $x, y \in Hom(\mathbb{Z}^d, \mathcal{H})$ which differ at most on finitely many sites, there exists a sequence of pivots starting from x and ending at y.

Question:

When does $Hom(\mathbb{Z}^d, \mathcal{H})$ have the pivot property?

Examples: Bipartite-Dismantlable Graphs

- $u \sim_{\mathcal{H}} v$ denotes (u, v) is an edge in $\mathcal{H}$.



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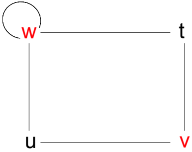

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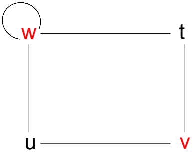

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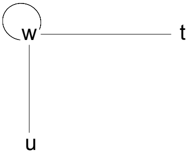

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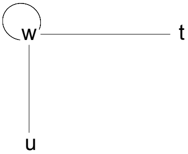

Theorem (Brightwell and Winkler '00)

If $\mathcal{H}$ folds into $\mathcal{H} \setminus \{v\}$ then $Hom(\mathbb{Z}^d, \mathcal{H})$ has the pivot property if and only if $Hom(\mathbb{Z}^d, \mathcal{H} \setminus \{v\})$ has the pivot property as well.

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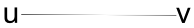

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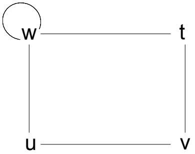

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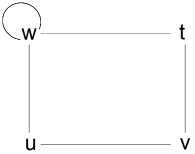

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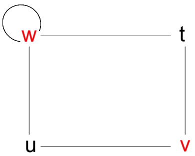

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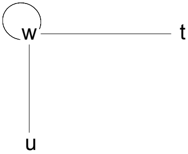

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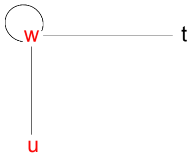

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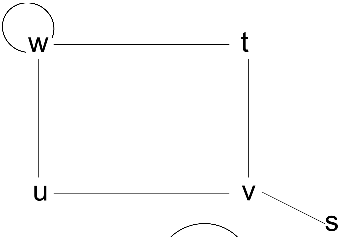
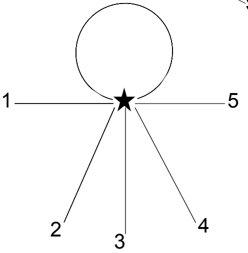
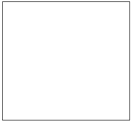
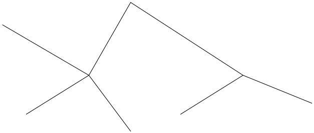




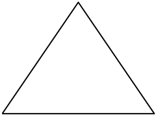


C_4

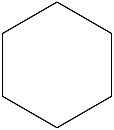


Trees

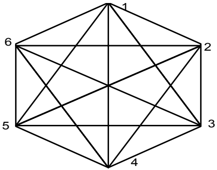
Examples: Not Bipartite-Dismantlable Graphs $\mathcal{H}$ for which $Hom(\mathbb{Z}^2, \mathcal{H})$ has the pivot property



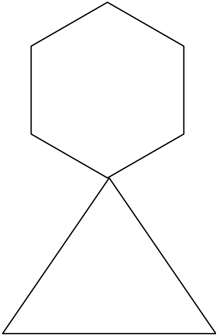
C_3



C_6

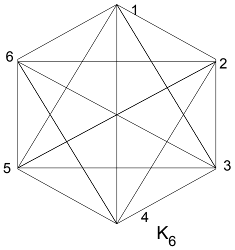


K_6



Examples: Complete Graphs

There are graphs $\mathcal{H}$ where no folding is possible but $Hom(\mathbb{Z}^d, \mathcal{H})$ still has the pivot property: Take $\mathcal{H} = K_6$ and $x \in Hom(\mathbb{Z}^2, K_6)$.

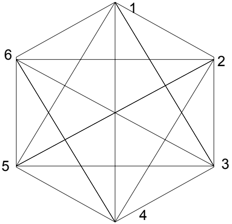


1	6	5	4	3	2	1	6
2	1	6	5	4	3	2	1
3	2	1	6	5	4	3	2
4	3	2	1	6	5	4	3
5	4	3	2	1	6	5	4
6	5	4	3	2	1	6	5
1	6	5	4	3	2	1	6
2	1	6	5	4	3	2	1

x

Examples: Complete Graphs

- The symbol at every site can be switched to a different admissible symbol.



K_6

1	6	5	4	3	2	1	6
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3	2	1	6	5	4	3	2
4	3	2	1	6	5	4	3
5	4	3	2	1	6	5	4
6	5	4	3	2	1	6	5
1	6	5	4	3	2	1	6
2	1	6	5	4	3	2	1

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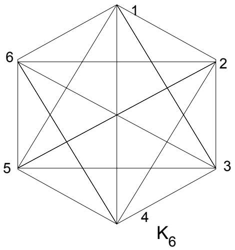
- Replace every appearance of 6 by some other admissible symbol one site at a time.



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6	5	4	3	2	1	6	5
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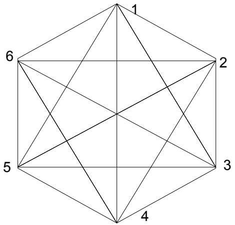

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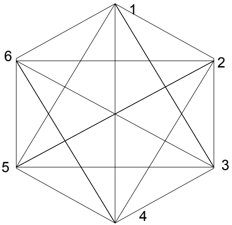

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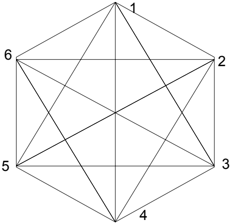

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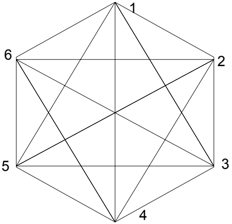

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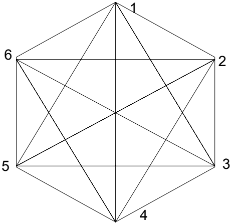

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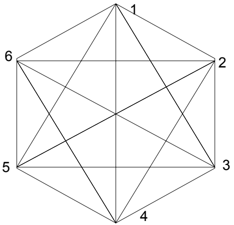
- The symbol at every site can be switched to a different admissible symbol.
- Replace every appearance of 6 by some other admissible symbol one site at a time.
- Now place 6 at every even position and finally 1 at every odd position to get a checkerboard pattern in 1's and 6's.



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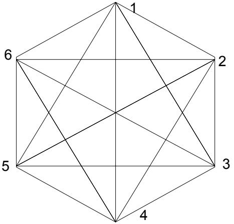

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6	2	6	2	6	4	6	2
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6	4	6	2	6	2	6	4
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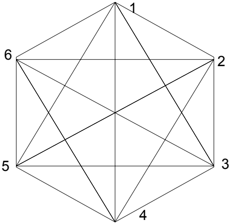


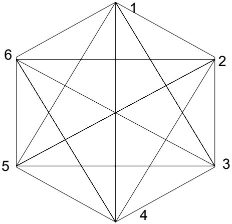
K_6

6	4	6	4	6	2	6	2
2	6	2	6	4	6	2	6
6	2	6	2	6	4	6	2
4	6	2	6	2	6	4	6
6	4	6	2	6	2	6	4
2	6	4	6	2	6	2	6
6	2	6	4	6	2	6	2
2	6	2	6	4	6	2	6

x

Examples: Complete Graphs

- The symbol at every site can be switched to a different admissible symbol.
- Replace every appearance of 6 by some other admissible symbol one site at a time.
- Now place 6 at every even position and finally 1 at every odd position to get a checkerboard pattern in 1's and 6's.



K_6

6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6

x

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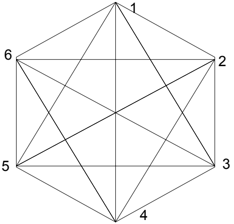

K_6

6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
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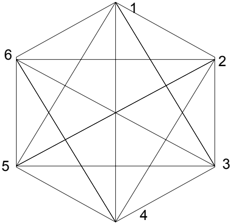
- Now place 6 at every even position and finally 1 at every odd position to get a checkerboard pattern in 1's and 6's.
- We can do this for any configuration $x \in Hom(\mathbb{Z}^2, K_6)$. Thus it has the pivot property.



K_6

6	1	6	1	6	1	6	1
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6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6
6	1	6	1	6	1	6	1
1	6	1	6	1	6	1	6

x

Examples: Complete Graphs

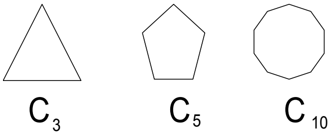
This can be further generalised to prove

Theorem (Well known)

$Hom(\mathbb{Z}^d, K_r)$ has the pivot property for all $r \geq 2d + 2$.

n-cycles

- C_n denotes the n-cycle with vertices $0, 1, 2, \ldots, n-1$.



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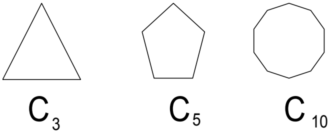
Theorem (Chandgotia, Meyerovitch '13)

$Hom(\mathbb{Z}^d, C_n)$ has the pivot property for all $n \neq 4$.



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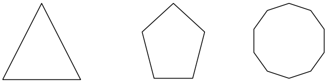
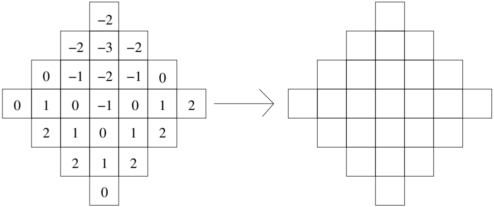
The result was well known for $n = 3$ and this was quite a simple generalisation.



C_3 C_5 C_{10}

$Hom(\mathbb{Z}^d, C_3)$- the 3-coloured chessboard

- A height function is an element of $Hom(\mathbb{Z}^d, \mathbb{Z})$.



Height Function

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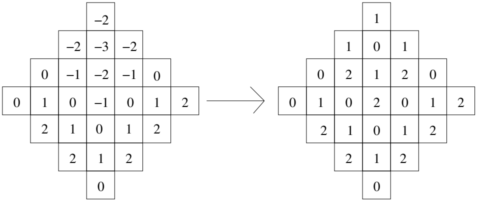
- If h is a height function then $h \mod 3$ is an element of $Hom(\mathbb{Z}^d, C_3)$.



Height Function   Element of Hom(Z^2,C$_3$)

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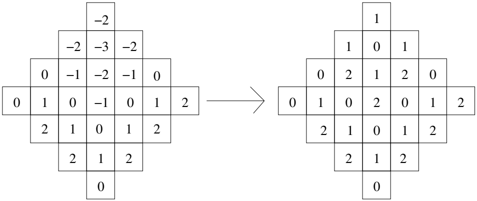

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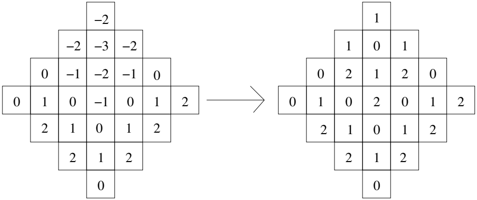
- It is sufficient to prove the pivot property for $Hom(\mathbb{Z}^d, \mathbb{Z})$.



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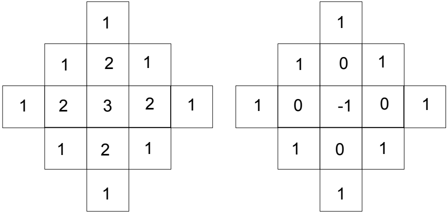

h k

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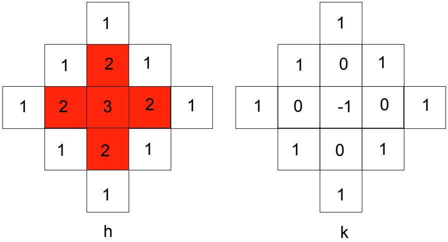

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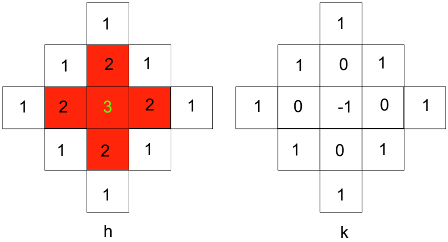

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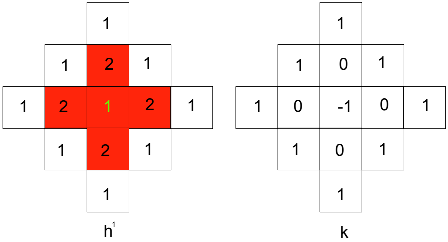

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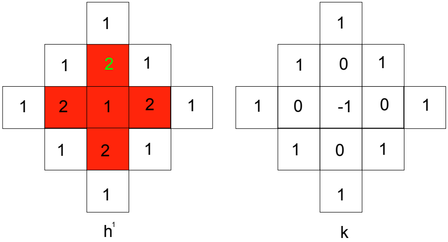

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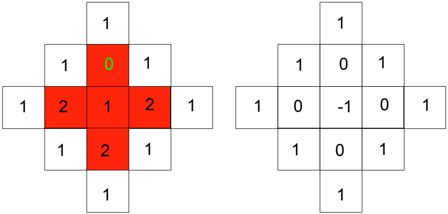

h^2 k

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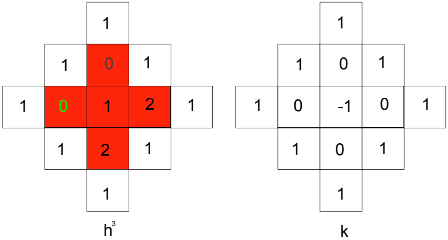

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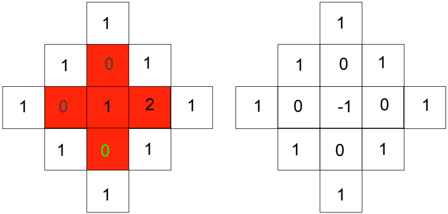

h^4 k

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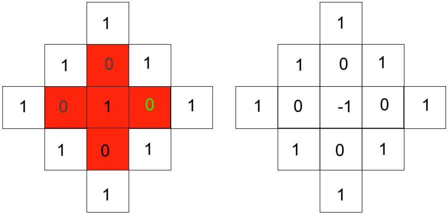

h^5 k

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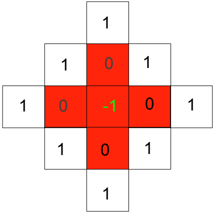
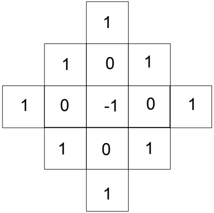


h^6=k



k

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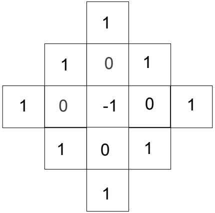
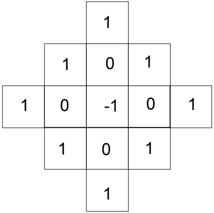


k



k

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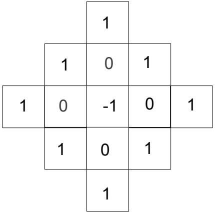
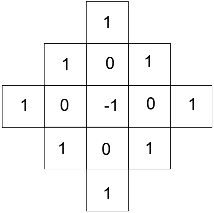


k



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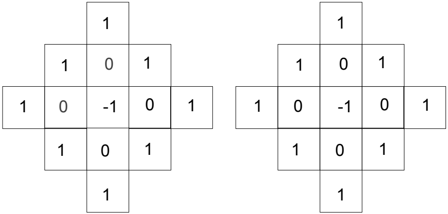
- Proceed similiarly with $k|_{F \setminus F_0}$.
- $Hom(\mathbb{Z}^d, \mathbb{Z})$ has the pivot property.



k k

Four-cycle free graphs

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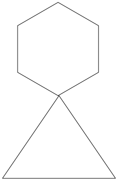
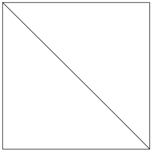


A four-cycle free graph



Not a four-cycle free graph

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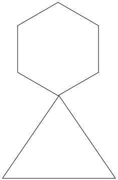

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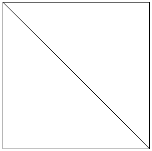

Not a four-cycle free graph

What generalises height functions for four-cycle free graphs?

Universal Covers

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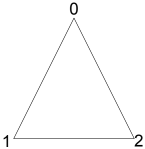
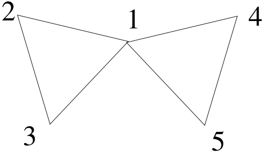
- Two such walks are adjacent if one extends the other by a single step.
- The universal cover of C_3 is $\mathbb{Z}$ (segments of the walks $0, 1, 2, 0, 1, 2, \ldots$ and $0, 2, 1, 0, 2, 1, \ldots$).



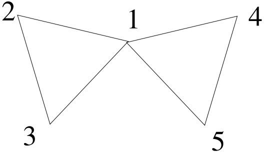
Universal Covers: An Example



$\mathcal{H}$

A Part of $E_{\mathcal{H}}$

Universal Covers: An Example



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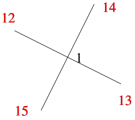

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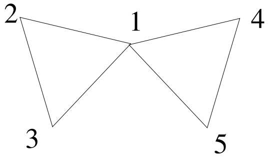
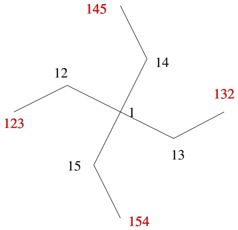
Universal Covers: An Example

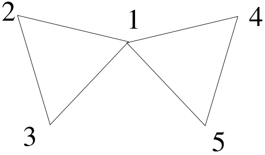


$\mathcal{H}$

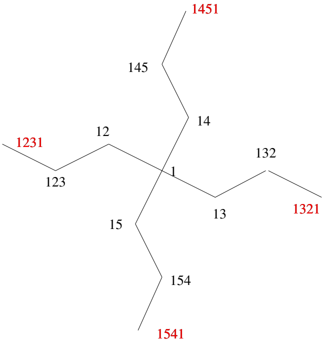


A Part of $E_{\mathcal{H}}$

Universal Covers: An Example

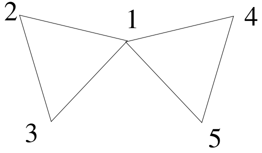


$\mathcal{H}$

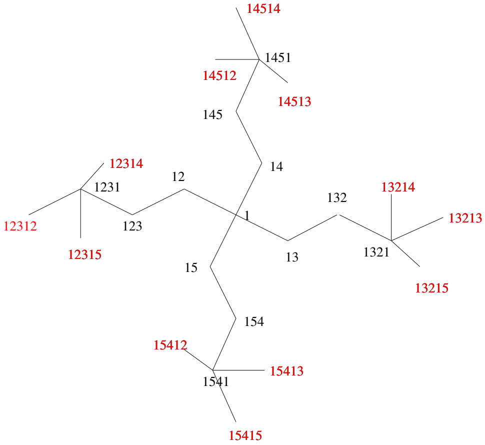


A Part of $E_{\mathcal{H}}$

Universal Covers: An Example



$\mathcal{H}$



A Part of $E_{\mathcal{H}}$

Four-cycle free graphs

This can be used to prove

Theorem (Chandgotia '14)

If $\mathcal{H}$ is a four-cycle free graph then $Hom(\mathbb{Z}^d, \mathcal{H})$ has the pivot property.

Are there homomorphism spaces which do not have the pivot property?

The generalised pivot property

$Hom(\mathbb{Z}^2, K_5)$ does not have the pivot property.

The generalised pivot property

$Hom(\mathbb{Z}^2, K_5)$ does not have the pivot property.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 3 & \boxed{4 \quad 5} & & 1 & 2 \\ 5 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{array}$$

The symbols in the box can be interchanged; but no individual symbol can be changed.

The generalised pivot property

$Hom(\mathbb{Z}^2, K_5)$ does not have the pivot property.

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

The symbols in the box can be interchanged; but no individual symbol can be changed. But it satisfies a more general property:

$Hom(\mathbb{Z}^d, \mathcal{H})$ has the generalised pivot property if there exists $P \subset \mathbb{Z}^d$ finite such that for all $x, y \in Hom(\mathbb{Z}^d, \mathcal{H})$ which differ at finitely many sites there exists a sequence $x = x^1, x^2, \dots, x^n = y \in Hom(\mathbb{Z}^d, \mathcal{H})$ such that x^i, x^{i+1} differ only on some translate of P.

$Hom(\mathbb{Z}^2, K_5)$

- Let $x, y \in Hom(\mathbb{Z}^2, K_5)$ differ exactly on $F \subset \mathbb{Z}^2$ where F is finite.

1	2	3	4	5
2	3	1	2	1
5	1	3	4	2
4	3	5	1	3
3	2	1	5	4

x

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

- Let $x, y \in Hom(\mathbb{Z}^2, K_5)$ differ exactly on $F \subset \mathbb{Z}^2$ where F is finite.

1	2	3	4	5
2	3	1	2	1
5	1	3	4	2
4	3	5	1	3
3	2	1	5	4

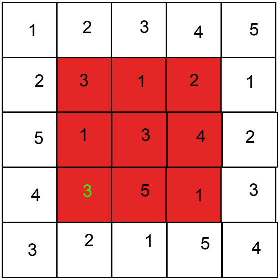
x

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

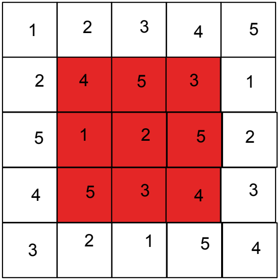
y

$Hom(\mathbb{Z}^2, K_5)$

- Let $x, y \in Hom(\mathbb{Z}^2, K_5)$ differ exactly on $F \subset \mathbb{Z}^2$ where F is finite.
- Choose the southwest-most site $\vec{i} \in F$. We want to change $x_{\vec{i}}$ to $y_{\vec{i}}$.



x



y

$Hom(\mathbb{Z}^2, K_5)$

- Let $x, y \in Hom(\mathbb{Z}^2, K_5)$ differ exactly on $F \subset \mathbb{Z}^2$ where F is finite.
- Choose the southwest-most site $\vec{i} \in F$. We want to change $x_{\vec{i}}$ to $y_{\vec{i}}$.
- Remove $x_{\vec{i}}, x_{\vec{i}+\vec{e}_1}, x_{\vec{i}+\vec{e}_2}$.

1	2	3	4	5
2	3	1	2	1
5	1	3	4	2
4	3	5	1	3
3	2	1	5	4

x

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

x

$Hom(\mathbb{Z}^2, K_5)$

- Let $x, y \in Hom(\mathbb{Z}^2, K_5)$ differ exactly on $F \subset \mathbb{Z}^2$ where F is finite.
- Choose the southwest-most site $\vec{i} \in F$. We want to change $x_{\vec{i}}$ to $y_{\vec{i}}$.
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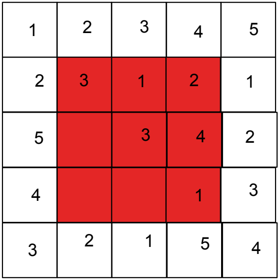
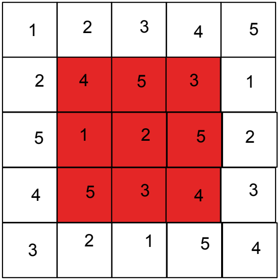


x



y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.

1	2	3	4	5
2	3	1	2	1
5		3	4	2
4	5		1	3
3	2	1	5	4

x

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
- The sites $\vec{i} + \vec{e}_1$ and $\vec{i} + \vec{e}_2$ are surrounded by four colours.

1	2	3	4	5
2	3	1	2	1
5		3	4	2
4	5		1	3
3	2	1	5	4

x

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
- The sites $\vec{i} + \vec{e}_1$ and $\vec{i} + \vec{e}_2$ are surrounded by four colours.
- We can always fill them in with a colour to get a valid configuration in $Hom(\mathbb{Z}^2, K_5)$.

1	2	3	4	5
2	3	1	2	1
5	1	3	4	2
4	5	2	1	3
3	2	1	5	4

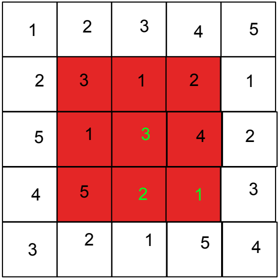
x^1

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

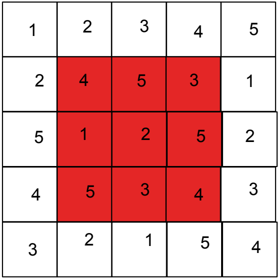
y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
- The sites $\vec{i} + \vec{e}_1$ and $\vec{i} + \vec{e}_2$ are surrounded by four colours.
- We can always fill them in with a colour to get a valid configuration in $Hom(\mathbb{Z}^2, K_5)$.
- Iterate. This proves that $Hom(\mathbb{Z}^2, K_5)$ has the generalised pivot property for the shape $P = \{\vec{0}, \vec{e}_1, \vec{e}_2\}$.



x^1



y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
- The sites $\vec{i} + \vec{e}_1$ and $\vec{i} + \vec{e}_2$ are surrounded by four colours.
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1	2	3	4	5
2	3	1	2	1
5	1	2	4	2
4	5	3	1	3
3	2	1	5	4

x^2

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
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1	2	3	4	5
2	3	1	2	1
5	1	2	4	2
4	5	3	1	3
3	2	1	5	4

x^1

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
- The sites $\vec{i} + \vec{e}_1$ and $\vec{i} + \vec{e}_2$ are surrounded by four colours.
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1	2	3	4	5
2	3	1	2	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

x^3

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
- The sites $\vec{i} + \vec{e}_1$ and $\vec{i} + \vec{e}_2$ are surrounded by four colours.
- We can always fill them in with a colour to get a valid configuration in $Hom(\mathbb{Z}^2, K_5)$.
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1	2	3	4	5
2	4	5	2	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

x^4

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

- Place $y_{\vec{i}}$ at the $\vec{i}$ site.
- The sites $\vec{i} + \vec{e}_1$ and $\vec{i} + \vec{e}_2$ are surrounded by four colours.
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1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

$x^5 = y$

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

$Hom(\mathbb{Z}^2, K_5)$

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1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

$x^5 = y$

1	2	3	4	5
2	4	5	3	1
5	1	2	5	2
4	5	3	4	3
3	2	1	5	4

y

Single-site Fillability

- $Hom(\mathbb{Z}^d, \mathcal{H})$ is single-site fillable if for $v_1, v_2, \ldots, v_{2d} \in \mathcal{H}$ there exists $v \in \mathcal{H}$ such that $v_i \sim_{\mathcal{H}} v$ for all $1 \leq i \leq 2d$.

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Theorem (Briceño '14)

If $Hom(\mathbb{Z}^d, \mathcal{H})$ is single-site fillable then it has the generalised pivot property.

Summary:

$Hom(\mathbb{Z}^d, \mathcal{H})$ has the pivot property if:

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Summary:

$Hom(\mathbb{Z}^d, \mathcal{H})$ has the pivot property if:

- $\mathcal{H}$ is bipartite-dismantlable. (Brightwell and Winkler '00)
- $\mathcal{H} = K_r$ where K_r is the complete graph on r vertices and $r \geq 2d + 2$. (well-known)
- $\mathcal{H}$ is four-cycle free. (Chandgotia '14)

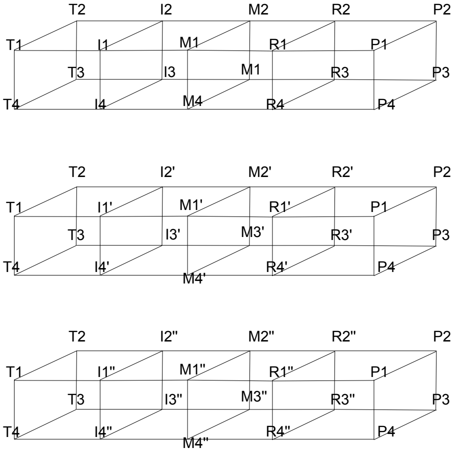
Summary:

$Hom(\mathbb{Z}^d, \mathcal{H})$ has the pivot property if:

- $\mathcal{H}$ is bipartite-dismantlable. (Brightwell and Winkler '00)
- $\mathcal{H} = K_r$ where K_r is the complete graph on r vertices and $r \geq 2d + 2$. (well-known)
- $\mathcal{H}$ is four-cycle free. (Chandgotia '14)
- $Hom(\mathbb{Z}^2, K_4)$, $Hom(\mathbb{Z}^2, K_5)$ do not have the pivot property but have the generalised pivot property (Briceño '14).

Theorem (Austin '16)

There is a graph $\mathcal{H}$ for which the space $Hom(\mathbb{Z}^2, \mathcal{H})$ does not have the generalised pivot property.



Question: Is the pivot property/generalised pivot property decidable for $Hom(\mathbb{Z}^d, \mathcal{H})$?

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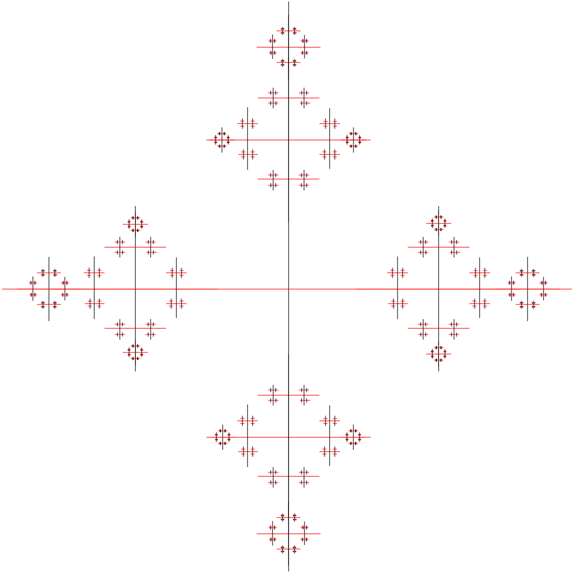
Question: How do we sample a random graph homomorphism in the absence of the generalised pivot property?



Thank You!