PIMS TALK, VANCOUVER - 2013

— joint work with Tom Meyerovitch.

All measures will be shift-invariant probability measures.

A Markov random field (MRF) is a probability measure on $A^{\mathbb{Z}^d}$ such that for all $A, B \subset \mathbb{Z}^d$ finite satisfying $\partial A \subset B \subset A^c$, $a \in \mathcal{A}^A$, $b \in \mathcal{A}^B$

$$\mu([a]_A \mid [b]_B) = \mu([a]_A \mid [b]_{\partial A})$$

outer boundary.

The collection of these objects is called a specification.

(need not correspond to a measure).

Question: Given supp(μ) can there exist a finite description of the specification for any MRF with support μ?

A nearest neighbour interaction is a function $V : A^{\square, \boxminus} \to \mathbb{R}$.

A Gibbs state is an MRF such that

$$\mu([x]_A \mid [x]_{\partial A}) = \frac{\sum_{C \in \square, \boxminus} V(x|_C)}{Z_{x|_{\partial A}}} \quad \text{for all } x \in \text{supp}(\mu)$$

topological support

Hammersley-Clifford Theorem: If supp(μ) has a safe symbol the

μ is an MRF $\Leftrightarrow$ μ is Gibbs with some nearest neighbour interaction (say) This does not hold without safe symbol. There constructions by Mussowis in ~~tdi~~ finite graphs.

~~μ is a probability shift-invariant measure~~

~~Note: $\Rightarrow$ supp(μ) is a shift~~

Note: μ is a MRF $\Rightarrow$ supp(μ) is a Shift space

(say) Further is true. It is a topological Markov field.

X n.n. SFT.

$\Delta_X = \{(x,y) \mid x$ and y differ at finitely many sites$\}$.

(say) Let us reparametrise the specifications.

Markov Cocycles X – TMF

$$c : \Delta_X \to \mathbb{R}$$

such that

- $c(x,y) = c(x,z) + c(z,y)$
- $c(x,y)$ is a function of $x|_{F \cup \partial F}, y|_{F \cup \partial F}$ where $F = \{i \mid x(i) \neq y(i)\}$.
- shift invariant.

Gibbs Cocycle: Markov Cocycle + there exists [Schmidt & Peterson] interaction V such that

$$c(x,y) = \sum_{C \in \text{II}} V(x|_C) - V(y|_C)$$

$\boxplus$

1-1 correspondence between Markov Cocycles and specifications. If μ is an MRF

$$c(x,y) = \log \frac{\mu([x]_F \mid [x]_{\partial F})}{\mu([y]_F \mid [y]_{\partial F})}$$ where

x, y differ at F is a Markov Cocycle.

Say (Radon Nikodym Cocycles)

finite dimensional — $\mathcal{G}_X$ — Gibbs Cocycles → vector space

$\cap$

M_X — Markov cocycles.

finite description corresponds to M_X finite

Stronger Version of Hammersley Clifford
X has a safe symbol. Then
$M_X = \mathcal{G}_X$.

Pivot property X has pivot property if for all $(x,y) \in \Delta_X$ there exists

$x = x_1, x_2, x_3 \cdots x_n = y$ such that
x_i, x_{i+1} differ ~~at~~ at a single site

Note. Then $c(x,y) = \cancel{c(x} \sum_{i=1}^{n-1} c(x_i, x_{i+1})$
— M_X finite dim

<u>Main Examples</u> X – 3 coloured checkerboard

$$\begin{array}{ccc} & 2 & \\ 2 & 1 & 2 \\ & 2 & \end{array} \rightsquigarrow \begin{array}{ccc} & 2 & \\ 2 & 3 & 2 \\ & 2 & \end{array} \qquad \begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array} \rightsquigarrow \begin{array}{ccc} & 1 & \\ 1 & 3 & 1 \\ & 1 & \end{array}$$

$$\begin{array}{ccc} & 3 & \\ 3 & 2 & 3 \\ & 3 & \end{array} \longrightarrow \begin{array}{ccc} & 3 & \\ 3 & 1 & 3 \\ & 3 & \end{array}$$

$\dim(M_X) = 3$

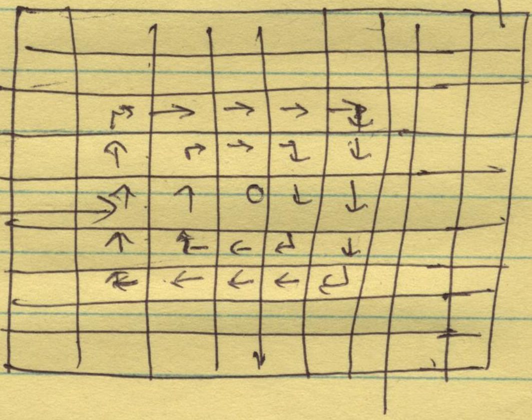
$\dim(G_X) = 2$

But every Markov random field is Gibbs.

Is the pivot property important?

X: Close ~~cousting~~ cousin of checkerboard island shift

→ blanks



squares →

X: Allowed blocks all 2×1, 1×2 blocks.

- topologically transitive.
- Square of various sizes and two distinct colours
- $\dim(M_X) = \infty$.