joint work – Jon Aaronson, M. Bromberg. (UCLA). 1

Recall the Central Limit Theorem $\{X_i\}$ – iid mean μ, variance σ^2

Let $S_n = \sum_{i=1}^{n} X_i$ then $\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{\text{dist.}} N(0,1)$.

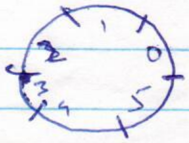
(The process $\{X_i\}$ carries a lot of entropy) – say

α – Quadratic irrational.

$\mathbb{T}$ – $[0,1)$ – circle.

$R_\alpha : \mathbb{T} \to \mathbb{T}$ given by $R_\alpha(x) := x + \alpha \mod 1$.

Features: 0 entropy, deterministic.



Let $\Phi : \mathbb{Z}/Q\mathbb{Z} \to \mathbb{R}^d$, • $\sum_{k \in \mathbb{Z}/Q\mathbb{Z}} \Phi(k) = 0$.

$Q = 6$

• $\mathrm{Span}(\mathrm{Img}(\Phi(\cdot))) = \mathbb{R}^d$.

Consider $\varphi : \mathbb{T} \to \mathbb{R}^d$ given by
$\varphi(x) := \Phi(\lfloor Qx \rfloor) := \Phi(k)$ if $\frac{k-1}{Q} \le x < \frac{k}{Q}$

Want to study $\varphi(0)$, $\varphi(\alpha) + \varphi(0)$, $\varphi(2\alpha) + \varphi(\alpha) + \varphi(0)$ . . . .

say (• orbit of a single point • in ergodic theory usually ~~orbit of~~ statement about almost every point.)

Ergodic theorem implies $\frac{1}{n} \sum_{t=0}^{n-1} \varphi(t\alpha) \to \int_{\mathbb{T}} \phi \, d\mu = 0$ Lebesgue measure

But what is the rate?

Let $\varphi_n(x) = \sum_{t=0}^{n-1} \varphi(x + t\alpha)$

• 1 2 3 . n.

S_n is a random variable.

Choose t uniformly, which chooses uniformly among $\varphi_t(0)$ $1 \le t \le n$

Theorem 1: $\exists\, \mu \in \mathbb{R}^d$ and l_k growing exponentially. such that

$$\frac{S_{l_k} - k\mu}{\sqrt{k}} \xrightarrow{\text{dist.}} N(0, \Sigma) \text{ for some } \Sigma.$$

- Aaronson Keane '82: α - Quadratic $\varphi = \chi_{[0,\frac{1}{2})} - \frac{1}{2}$
- Beck '10: - - - $\varphi = \chi_{[0,\beta)} - \beta$; β rational. say (on the full sequence).

~~Sarig~~ Dolgopyat, Sarig '16: ~~φ~~ α - Quadratic, $\varphi = \chi_{[0,\beta)} - \beta$. β rational say (~~different~~ for all initial points).

Aaronson, Bromberg, Nakada '16 α - Badly approximable $\varphi : \chi_{[0,\frac{1}{2})} - \frac{1}{2}$.

— x —

μ & σ → interpreted by Beck int terms of sol'ns to the Pell's Equation.

→ Hubert, Hubert and Weiss in terms of translation flows.

Before we try to answer this.

Qn: How do we know φ_n grows at all?

Let $\Pi = \langle \Phi(\mathbb{Z}/Q\mathbb{Z}) \rangle$ say [Closed group generated by ... ~~$\mathbb{Z}/Q$~~ ...]

Consider the transformation: $(R_\alpha)_\varphi : \mathbb{T} \times \Pi \circlearrowleft$

$$(R_\alpha)_\varphi (x, s) = (x + \alpha,\ s + \varphi(x)).$$

Note: $((R_\alpha)_\varphi)^n (x, s) = (x + n\alpha,\ s + \underbrace{\varphi_n(x)}_{})$ ↪ ergodicity of φ_n measures spread.

Thm 2: ~~For $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ and Q is prime.~~
~~or Almost every α~~ There exists a set of measure 1; $\mathcal{E}_Q \subset \mathbb{T}$
$(R_\alpha)_\varphi$ is ergodic if Q is prime. & $\alpha \in \mathcal{E}_Q$.

~~Open~~ Qn: What if $Q = 6$ and $\alpha \in \mathbb{T} \setminus (\mathcal{E}_Q \cup \mathbb{Q})$?

$\chi_{[0,\frac{1}{2})} - \frac{1}{2}$: Schmidt ?, Conze & Keane '76. ($\alpha = \frac{\sqrt{5}-1}{4}$, ---)

$\chi_{[0,\beta)} - \beta$. β-rational, $(1, \alpha, \beta)$ rationally independent.

Stewart '81, --- Oren '83. (Conditions of β, full result).

[illegible]

How is something like this proved? $\| \ \|$ → distance from closest integer.

Essential value ~~like~~ condition: (Schmidt / Orem)[1] $a \in \Gamma$ is an essential value if $\exists$ const

$n_k \to \infty$ s.t. $\mu(\varphi_{n_k} \doteq a) > c$.

$\varphi_{n_k} \Longleftrightarrow \ \cdot \ \|\alpha n_k\| \to 0$

⇒ Essential values form a closed group.

⇒ c is essential value and f is $(R_\alpha)_\varphi$-invariant

$\Rightarrow f(x, r + c) = f(x, r)$ a.e.

$\Longrightarrow$ essential values $= \Gamma \Longleftrightarrow (R_\alpha)_\varphi$ is ergodic.

$\varphi = \chi_{[0, \frac{1}{2})} - \frac{1}{2}$

Need to prove: $\frac{1}{2}$ is an essential value.

Continued fractions: $\alpha = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \ddots}}$. Let $\dfrac{p_k}{q_k} = \cfrac{1}{n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \ddots + \cfrac{1}{n_k}}}}$

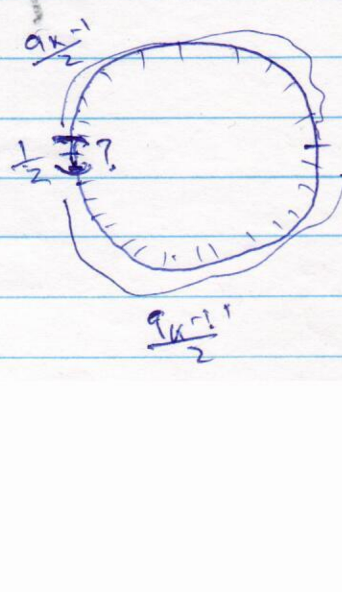
Facts:
(a) p_k, q_k are prime to each other.
(b) $\|\alpha - \frac{p_k}{q_k}\| < \frac{1}{q_k^2}$.

p_k, q_k prime to each other, to

$$\left\{\frac{t p_k}{q_k}\right\}_{t=0}^{q_k - 1} = \left\{\frac{t}{q_k}\right\}_{t=0}^{q_k - 1}$$

Fix $x \in \Pi$

(b) ⇒ each interval $\left[\frac{r}{q_k}, \frac{r+1}{q_k}\right)$ contains exactly one $\{x + t\alpha\}_{0 \le t < q_k}$.



(c) q_k can be chosen ~~large~~ odd. (and large).

~~⇒ $\frac{q_k - 1}{2}$ of $\{t\alpha\} \in [0, \frac{1}{2})$~~

$\left| \# q_k : \{x + t\alpha\} \text{ in } [0, \tfrac{1}{2}) - \# \{x + t\alpha\} \text{ in } [\tfrac{1}{2}, 1) \right| = 1$

⇒ Denjoy–Koksma ...

$\therefore\ \varphi_{q_k} = \frac{1}{2}$ or $-\frac{1}{2}$. Thus $\frac{1}{2}$ or $-\frac{1}{2}$ is an essential value.

— x —

(Ergodicity is not a strong condition for infinite m.p.t) — Say.

$\Rightarrow$ ~~__~~ $f \in L^1(\mu)$ $\xrightarrow{\text{infinite}}$ m.pt. (X, μ, T)

$$\forall a_n \uparrow \infty \quad \overline{\lim_{n\to\infty}} \frac{1}{a_n}\sum_{t=0}^{n-1} f(T^t x) = \infty \ \cancel{\text{or } 0}$$

or.

$$\underline{\lim_{n\to\infty}} \frac{1}{a_n}\sum_{t=0}^{n-1} f(T^t x) = 0$$

(BRE)

Bounded rational ergodicity (something like an ergodic Theorem holds).

$\exists a_n \uparrow \infty$ ~~and $a_n \to 1$~~

$$\overline{\lim_{n\to\infty}} \frac{1}{a_n} \sum_{t=0}^{n-1} f(T^t x) = \int f\, d\mu. \text{ a.e.}$$

Defn:

(X, μ, T) is (B.R.E) if $\exists A, 0<\mu(A)<\infty$ s.t

$$\sup_{n\geq 1}\ \operatorname{ess.-sup}_{x\in X} \left| \frac{\sum_{t=0}^{n-1} \chi_A(T^t x)}{\sum_{t=0}^{n-1} \mu(A\cap T^{-t}A)} \right| < \infty$$

$\downarrow$ ~~__~~ saying $\sum_{t=0}^{n-1} \chi_A$ visits to 0 of almost every point $x \in \mathbb{T}$ is $\asymp \frac{n}{(\log n)^{d/2}}$

Thm 3: ~~__~~ $\langle \Phi(\mathbb{Z}/q_\mathbb{Z}) \rangle$ (α- Quadratic.)

$= \Gamma \subset \mathbb{R}^d$ is discrete.

$(R_\alpha)_\varphi$ is b.r.e.; $A = \mathbb{T}\times\{\vec{0}\}$

Qu: What if $\langle \Phi(\mathbb{Z}/q_\mathbb{Z})\rangle = \mathbb{R}^d$?

~~Need~~

For this we need to understand visits to $\mathbb{T} \times B(0, \varepsilon)$.

We can't use: $\varphi_n(x) = 0$ and $\varphi_m(T^n x) = 0$

$\Rightarrow \varphi_{n+m}(x) = 0$

But $\varphi_n(x) \in, \varphi_m(T^n x) \in B(0,\varepsilon) \not\Rightarrow \varphi_{n+m}(x) \in B(0,\varepsilon)$

Why α - Quadratic?

We want to look at $\varphi(0) + \varphi(\alpha) + \dots \varphi((n-1)\alpha)$

Want to understand $\varphi(t\alpha)$. Let $\alpha = \frac{\beta + P}{Q}$ $P \in \{0, 1, \dots q-1\}$, $0 < \beta < 1$.

Recall:

$$\varphi(t\alpha) = \Phi\left(\left\lfloor t \frac{P+\beta}{Q} Q \right\rfloor\right)$$

$$= \begin{cases} \Phi(tP + \lfloor (t-1)\beta \rfloor + 1 & \text{if } (t-1)\beta \in [1-\beta, 1) \\ \Phi(tP + \lfloor (t-1)\beta \rfloor) & \text{o.w.} \end{cases}$$

Qn: When does $(t-1)\beta \in [1-\beta, 1)$? $\rightsquigarrow$ Sturmian Sequences.

$$\beta = \cfrac{1}{n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 \dots}}} \doteq [n_1, n_2, n_3 \dots]$$

β quadratic irr. iff $n_1 n_2 \dots$ is eventually periodic

The sequence $(\chi_{[1-\beta,1)}(t\beta))_{t=0}^{n-1}$ is given by:

$b_0 = (0)$ $b_0(1) = 1$

$b_{k+1}(0) = b_k(0)^{\odot(n_{k+1}-1)} \odot b_k(1)$

$b_{k+1}(1) = b_k(0)^{\odot(n_{k+1}-2)} \odot b_k(1)$

Let $l_k := |b_k(0)|$

$$(\chi_{[1-\beta,1)}(t\beta))_{t=0}^{l_k - 1} = b_k(0).$$

6

n_k's eventually periodic.

$\rightsquigarrow$ transitions of b_k's eventually periodic.

$\rightsquigarrow (\varphi(t\alpha))_{0 \le t \le l_k - 1}$ $\rightsquigarrow$ eventually periodic. transitions

$\rightsquigarrow (\varphi_t(0))_{0 \le t \le l_k - 1}$ is a " " (transition)

$\boxed{\times \text{ not.}}$

~~Following carefully, we get.~~

<u>Recall</u>: S_{l_k} is a r.v. ~~which takes~~

$= \varphi_t(0)$ $1 \le t \le l_k$ w.p. $\frac{1}{l_k}$.

- (X_k) is $(\mathbb{R}^d)^{\{0,1\} \times Q}$ valued r.v.

$(X_k)^{(0,0)} = S_{l_k}$.

and $X_{k+1} = \underbrace{A_{k+1} X_k + (C_{k+1} + k D_{k+1})}_{\text{Random Affine Transformation (RAT)}}$

where $A_{k+1}, C_{k+1}, D_{k+1}$ independent $\xrightarrow[\text{converging in distribution to}]{} A_\infty, C_\infty, D_\infty$

- $D_\infty \to$ Contributes the mean

We prove a Central limit theorem for such transformation

$\rightarrow$ Central limit Theorem for S_{l_k} and proves Theorem 1.