1

$$\cancel{\lim_{n\to\infty} \frac{1}{a_n} \sum \mathbb{1}_A \circ T^k = \mu(A)}.$$

Talk for Weizmann j.w. Aaronson, Bromberg

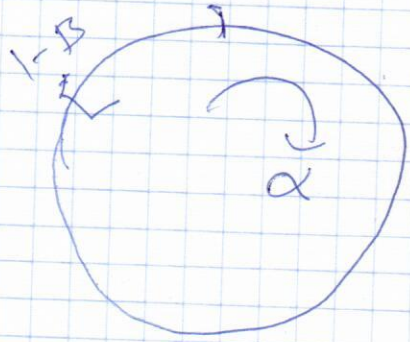
(Question of Veech)

$\mathbb{T} = [0,1)$ addition mod. 1

α - irrational.

$$S_n^\beta(x) = \underbrace{\sum_{i=0}^{n-1} \mathbb{1}_{[1-\beta,1)}(x+i\alpha)}$$

visits of $x+i\alpha$ to interval $[1-\beta, 1]$



Qu: Is $\{S_n^\beta(x) - n\beta\}_{n\in\mathbb{N}}$ dense in $\langle 1,\beta\rangle$ for a.e. x?

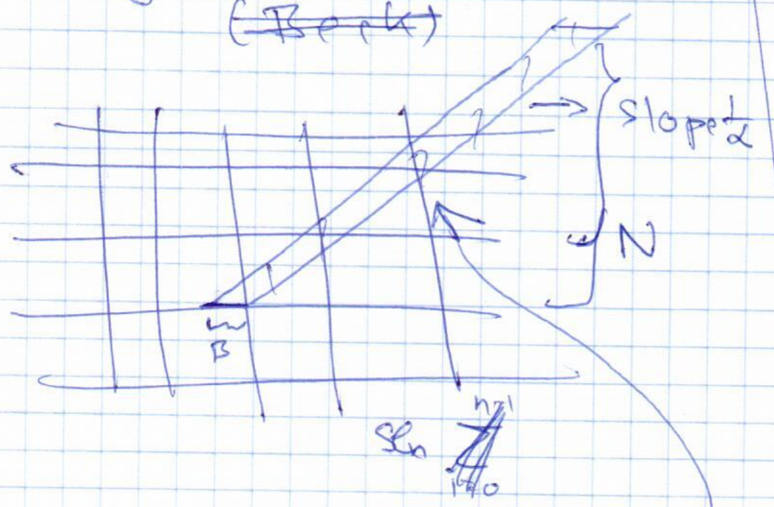
closed group generated by $1, \beta$

(not every x by Markley)

Why was Veech Interested?

Counting Lattice points.

(~~Beck~~)



$S_n^\beta(0)$ = # integer points in

Weyl' result says

$$\frac{1}{n} S_n^\beta(x) - \beta \longrightarrow 0$$

as $n \to \infty$

Want to understand finer asymptotics.

Hecke | Kesten. $\{S_n^\beta(x) - n\beta\}_{n=1}^\infty$ is bounded iff $\beta = \{k\alpha\}$ for some $k \in \mathbb{Z}$.

$\{\cdot\}$ → fractional part.

Let us see this in the case $\beta = \alpha$

$$S_n^\alpha(x) - n\alpha = \sum_{i=0}^{n-1} \left(\mathbb{1}_{[1-\alpha,1)}(x + i\alpha) - \alpha \right)$$

Let $f : \mathbb{T} \to \mathbb{R}$ given by $f(x) = -x$.

Then
$$f(x+\alpha) - f(x) = \begin{cases} -x - \alpha + x & 0 \le x < 1-\alpha \\ -x - \alpha + 1 + x & 1-\alpha \le x < 1 \end{cases}$$

(Telescope)
$$= \mathbb{1}_{[1-\alpha,1)}(x) - \alpha.$$

Thus $|S_n^\alpha(x) - n\alpha| \le 2\|f\|_\infty \le 2$

(Say)

$\left(\mathbb{1}_{[1-\alpha), 1)}(x) - \alpha \right)$ is coboundary where the cobounding function is f.

3

$$\left\{ S_n^{\beta}(x) - n\beta \right\}_{n=1}^{\infty}$$

$\boxed{\alpha \text{ quadratic}}$

Dense iff $\beta \notin \mathbb{Z}(\alpha)$

Oren: $\boxed{\text{ergodic.}}$

estimates on visits to 0 for a.e. starting point

Aaronson & Keane '79

$\boxed{\text{bounded rational}}$

$\boxed{\text{ergodicity}}$

$\boxed{\beta = \frac{1}{2}}$

Central Limit Theorem.

Aaronson & Keane '79

Beck, Dolgopyat & Sarig (2090) (2090)

$\boxed{\beta = \frac{1}{2}}$

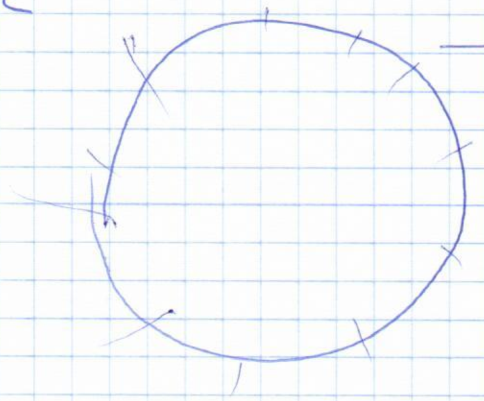
Computation of constants

$\boxed{\beta \text{ rational}}$

$\boxed{\beta \text{ irrational}}$

~~Cosi~~ Bromberg & Ulcigrai.

$Q \geq 2$



$\longrightarrow$ Label with $\mathbb{Z}/Q\mathbb{Z}$.

$\rightsquigarrow \mathbb{R}^d$. such that they add to 0.

Formally. $\Phi : \mathbb{Z}/_{Q\mathbb{Z}} \longrightarrow \mathbb{R}^d$ s.t.

$$\sum_{k \in \mathbb{Z}/_{Q\mathbb{Z}}} \Phi(k) = 0$$

$\varphi : \mathbb{T} \to \mathbb{R}^d$ given by

$$\varphi(x) = \Phi(\lfloor Qx \rfloor)$$

~~$(R_\alpha)_\varphi : \mathbb{T} \times \mathbb{R}^d \to \mathbb{T} \times \mathbb{R}^d$. (Haar measure)~~.

$\Gamma \subset \mathbb{R}^d$. closed group generated by (Φ)

~~$(R_\alpha)_\varphi$~~ $\longrightarrow$ think of $\mathbb{Z}^d / \mathbb{R}^d$.

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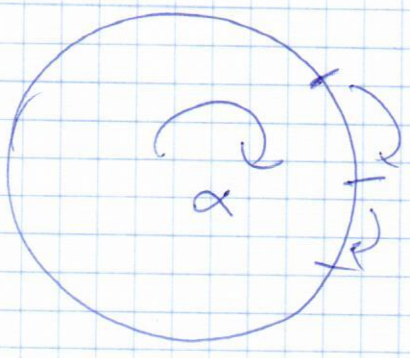
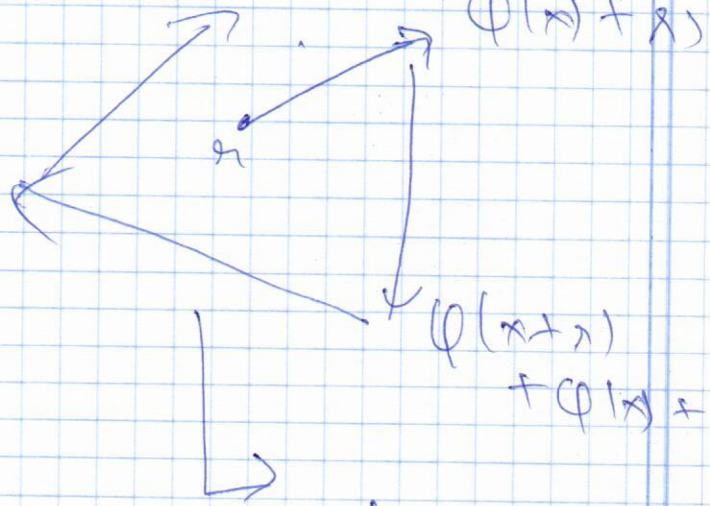
$(R_\alpha)_\varphi : \mathbb{T}\times\Gamma \longrightarrow \mathbb{T}\times\Gamma$ (Haar Measure)

$$(x, g) \longrightarrow (x+\alpha, g+\varphi(x))$$





deterministic walk!

- (X,μ) measure space.

We say that $T : (X,\mu) \to (X,\mu)$ is ergodic if $\forall A, B \subset X$, $\mu(A), \mu(B) > 0$ $\exists n$ s.t. $\mu(T^{-n}A \cap B) > 0$.

<u>Theorem</u>: $\exists A \subset \mathbb{T}$ of full measure s.t. $(R_\alpha)_\varphi$ is ergodic if

- Q is prime. α-irrational &
- $\alpha \in A$.

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Open: $Q = 0$ α is any irrational

∘ $\mathbb{T}$ replaced by $\mathbb{T}^d$

ergodic Theorem $\mu(X) = 1$.

(X, μ, T) ergodic $f \in L^1(\mu)$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} f(T^t x) = \int f\,d\mu \text{ a.e.}$$

Aaronson '77 $\mu(X) = \infty$ nothing like the ergodic theorem holds.

$\varphi : \mathbb{T} \to \mathbb{R}^d$

$$\varphi_n = \sum_{t=0}^{n-1} \varphi(x + t\alpha) \doteq \#\text{ visits to } 0$$
of interval

Aaronson $\Rightarrow$ $\forall a_n \uparrow \infty$ $\forall f \in L^1(\mu)$

either $\limsup \frac{1}{a_n} f_n = \infty$

or

$\liminf \frac{1}{a_n} f_n = 0$

$\boxed{\text{Thm 2}}$ If $\langle \Phi(\mathbb{Z}/q_n) \rangle$ is discrete α-quadratic

$\exists a_n \uparrow \infty$ st ~~$\limsup_n \frac{1}{a_n} \varphi_n \to \int \varphi\,d\mu = 0$~~

(7)

$$\frac{1}{a_n}\sum_{k=0}^{n-1} \mathbb{1}_{A \mathbb{T}\times\{\vec{0}\}}\left((R_\alpha)_\varphi^k (x, \vec{0}) \right) \longrightarrow 1$$

for a.e. $x \in \mathbb{T}$

Can replace by any L^1 function f same ann. $\int f \, d\mu$.

$$\boxed{\text{Open: } \langle \Phi \mid \mathbb{Z}/(q_?) \rangle \text{ is not discrete.}}$$

X_k = temporal statistics of the sequence $\varphi_1, \varphi_2, \ldots$

<u>Theorem 3</u>: If α - quadratic and $\mathrm{span}(\varphi(\mathbb{T})) = \mathbb{R}^d$.

$\exists\, l_k \uparrow \infty$, $\vec{\mu} \in \mathbb{R}^d$, $\Sigma \in M_{d\times d}(\mathbb{R})$ st.

$$\frac{X_{l_k} - k\vec{\mu}}{\sqrt{k}} \to N(0, \Sigma)$$

$\updownarrow$ Contrast

(Sum of iids . . .)

Open: • All sequences $\frac{X_k - \log k\, \vec{\mu}}{\sqrt{\log k}} \longrightarrow ?$

• Interpretation of $\vec{\mu}$ & Σ.

- $\mathbb{T} \longrightarrow \mathbb{T}^d$.

α - not quadratic irrational.

1) α is Badly approximable, $\varphi = \chi_{[0,\frac{1}{2})} - \frac{1}{2}$
then temporal CLT holds.

2) For a.e. α & $\varphi = \chi_{[0,\frac{1}{2})} - \frac{1}{2}$.
$\exists\, l_k \uparrow \infty$ & μ, σ s.t.

$$\frac{x_{l_k} - (\log k)\,\mu}{\sqrt{\log k}\;\sigma} \to U([0,1])$$

• Omiro Dolgopyat & every r. Equidistribution → Uniform distribution.

• Omiri & Dolgopyat : Anosov diff. for almost every point
limit point of temporal stats = all random variables

• Bretom & Denker: ∃ CLT for som L_+ function.
spatial statistics.

~~Keskin~~
~~Fayyad &~~

Keskin: spatial / spatial temporal $\lim_{n\to\infty} \mathbb{P}(m : w_n\langle k\alpha\rangle \le \alpha) \to ?$ $1 \le k < m$

$\lim_{n\to\infty} \mathbb{P}\{\xi : 0 \le \xi \le 1$ $\exists a, b.$ $m \le b \le m$ $|b\xi - a| \le \frac{\alpha}{b}\}$

multiple dimensions.

1 • boundedness of discrepancy, coboundary.
• changing both initial point & rotation → Cauchy distr.
discrepancy

Keskin: SDS • measure of ξ $N \le a_n \le cN$ exists

Fayyad, Dolgopyat, Vinogradov: Shrinking target problem. CLT
Dolgopyat & Fayyad → α, x random → Convex Body discrepancy
→ Boxes Cauchy distribution

Avila, Dolgopyat, Dusyen & Sarig → Linear $\varphi = \chi_{[0,\frac{1}{2}]} - \frac{1}{2}$