Some strange universality results Among Hom-shifts.

$\mathcal{G}$, $\mathcal{H}$ - undirected graphs.

finite

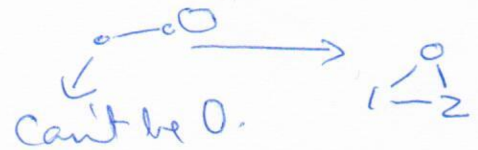
$x : \mathcal{G} \to \mathcal{H}$. Denote $x_i := x(i)$

$i \underset{\mathcal{H}}{\sim} j$ means i is adjacent to j in $\mathcal{G}$.

$Hom(\mathcal{G}, \mathcal{H}) = \{ x : \mathcal{G} \to \mathcal{H} \ : \ i \underset{\mathcal{G}}{\sim} j \Rightarrow x_i \underset{\mathcal{H}}{\sim} x_j \}$

~~$\mathcal{H}$~~ $Hom\big(\mathcal{G}$,  $\big)$ Colourings of $\mathcal{G}$ with 0, 1 and 2 where adjacent colours are distinct

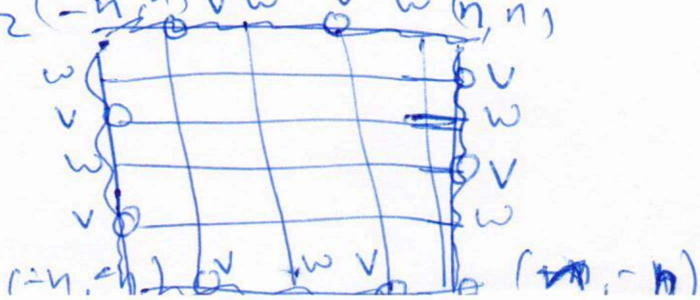
Say (no self loops.



can't be 0.
$0 \underset{\mathcal{H}}{\not\sim} 0$)

$Hom(\mathcal{G}, ⟲0 - 1)$ 0, 1 patterns where no two 1's are distinct.

$\mathcal{B}_n = \{-n, \dots, n\}^2 \subset \mathbb{Z}^2$



$\partial B_n \doteq \{(i,j) \in B_n \mid |i| \text{ or } |j| = n\}$. ②

odd — $|i|+|j|$ is odd

even — $|i|-|j|$ is even. $v, w \in H$

$$\text{Flat}_n = \left\{ x \in \text{Hom}(B_n, H) : x|_{\partial B_n, \text{odd}} = v,\ x|_{\partial B_n, \text{even}} = w \right\}.$$

Thm 1: (C., Peled '16) ~~P~~ Uniform probability on Flat_n

$\cancel{\frac{|\text{Flat}_n|}{|\text{Hom}(B_n, H)|}}$ $\quad \frac{\text{Flat}_n}{\cancel{|\text{Hom}(B_n, H)|}} \geq e^{-cn}$. (Say uniform probability - - -)

then $\mathbb{P}(\text{Flat}_n)$

This is sharp. C_∞ — ⓘ $\doteq$ no constraints

$$\frac{\text{Flat}_n}{|\text{Hom}(B_n, H)|} = \frac{|\text{Hom}(B_{n-1}, H)|}{|\text{Hom}(B_{n+1}, H)|} = \frac{1}{2^{\cancel{(2n+3)}}}$$

$$= \frac{2^{(2n-1)^2}}{2^{(2n+1)^2}} \sim e^{-cn} \text{ for some } c.$$

~~A – finite alphabet.~~

~~Now~~ ~~$(A^{\mathbb{Z}^2}, \sigma)$~~ Phenomena

Says (Constant configuration on the boundary has high probability).

Changing gears

$\mathcal{A}$ - finite alphabet.

$\sigma^{\vec{i}} : \mathcal{A}^{\mathbb{Z}^2} \longrightarrow \mathcal{A}^{\mathbb{Z}^2}$

$(\sigma^{\vec{i}}(x))_{\vec{j}} = x_{\vec{i}+\vec{j}}$.

Shift space: $X \subset \mathcal{A}^{\mathbb{Z}^2}$ - closed.
- invariant under σ

(X, σ) is a dynamical system.
$\hookrightarrow \mathbb{Z}^2$ action.

Examples: $Hom(\mathbb{Z}^2, H) =: X_H$.

Entropy $B \subset \mathbb{Z}^2$. $\mathcal{L}_B(X) := \{x|_B : x \in X\}$.

Top Entropy $h_{top}(X) := \lim_{n \to \infty} \log \frac{|\mathcal{L}_{B_n}(X)|}{|B_n|}$.

Sanity Check: $h_{top}(\mathcal{A}^{\mathbb{Z}^d}) = \lim_{n\to\infty} \log \frac{|\mathcal{A}|^{|B_n|}}{|B_n|}$

$= |\mathcal{A}|$

$h_{top}(X_{0-1}) = ??$

$h_{top}(X_{1\overset{0}{\angle}2}) = \log\left(\frac{8\sqrt{3}}{9}\right)$

Notice. $\cancel{\lim_{n\to\infty} \mathrm{Flat}_n}$

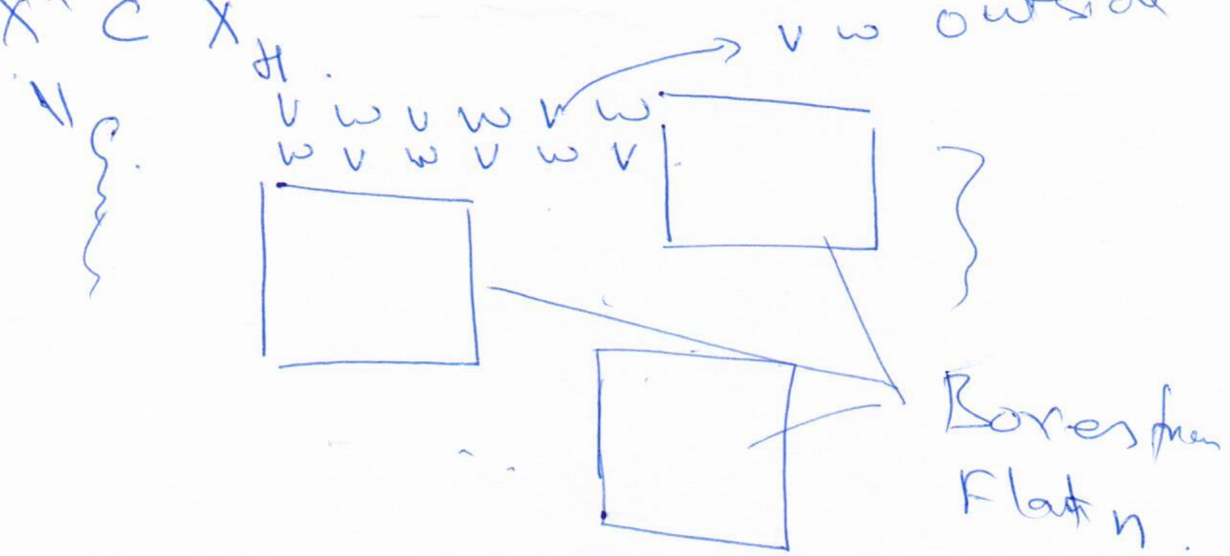
$$1 \geq \frac{|\mathrm{Flat}_n|}{|\mathrm{Hom}(B_n, H)|} = \frac{|\mathrm{Flat}_n|}{|\mathcal{L}_{B_n}(X_H)|} \geq e^{-cn}.$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log(\mathrm{Flat}_n)}{(2n+1)^2} - \lim_{n\to\infty} \frac{\log|\mathcal{L}_{B_n}(X_H)|}{(2n+1)^2} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\log(\mathrm{Flat}_n)}{(2n+1)^2} = h_{top}(X_H).$$

X^n be shift space.

$X^n \subset X_H$



~~Can Prove~~ **Thm 2:** $\lim_{n\to\infty} h_{top}(X^n) = h_{top}(X_H)$.

ϕ

(X, T) dynamical system. $T^{\vec{i}}: X \to X$, $\vec{i} \in \mathbb{Z}^2$.

μ probability measure

$T^{\vec{i}}\mu = \mu$.

(X, μ, T) – ~~prob pre~~ probability preserving transformation

⇝ Can define h_μ ⇝ measure-theoretic entropy.

(weighted version of topological entropy)

Variational principle

$$\sup_{T\mu = \mu} h_\mu = h_{top}(X).$$

~~Theorem~~

We say $(X, \mu, T) \simeq (Y, \nu, S)$

if $\exists f$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\ Y \arrow[r, "S"'] & Y \end{tikzcd}$$

and f is an isomorphism ~~of measure spaces~~ (X, μ) (Y, ν).

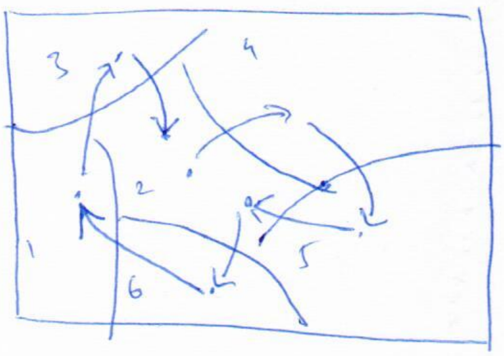
(X, T) is universal if $\forall (Y, \nu, S)$ ~~universe~~ ((+ technical assumption)

$h_\nu < h_{top}(X)$ $\exists \mu$ st

$$(X, \mu, T) \simeq (Y, \nu, S)$$

$(A^{\mathbb{Z}^d}, \mathcal{B}, \sigma)$ is <u>universal</u> (Kreiger '72, Rosenthal '88, Kammeyer '90)

$\exists$ partition



X

&

~~encodes points $\longrightarrow$ orbits~~

Points $\xleftrightarrow[\text{encoding}]{1-1}$ trajectories of points.

<u>Qn:</u> (Sahin & Robinson) '2000 Is $X_{\triangle}$ universal?

Thm 3: (X_H, σ^2) is universal.

& If H has a self-loops

(X_H, σ) is universal.

Conjecture: If H is not bipartite

(X_H, σ) is universal.

Say (H is bipartite $\rightsquigarrow$ best possible result)

X^n. v w v w v w
w v w v w v

(X, σ) is called SI
if $\exists n$ st
$\forall$ $A, B \subset \mathbb{Z}^?$, $d(A,B) \geq n$.
$a \in \mathcal{L}_A(X)$, $b \in \mathcal{L}_B(X)$, $\exists x \in X$ st
$x|_A = a$ & $x|_B = b$.

Thm (Robinson, & Şahin 2000) SI $\Rightarrow$ universal.

We can prove. $(X^{(n)}_{\mathbb{Z}^d}, \sigma^2)$ is SI

$\Rightarrow$ $(X^{(n)}, \sigma^2)$ is universal.

~~$\Rightarrow$~~ & $h_{top}(X^{(n)}) \to \mathcal{L}_{top}(X_H) \Rightarrow (X_H {}_{\mathbb{Z}^d}, \sigma^2)$ universal.

~~How to prove the Theo~~

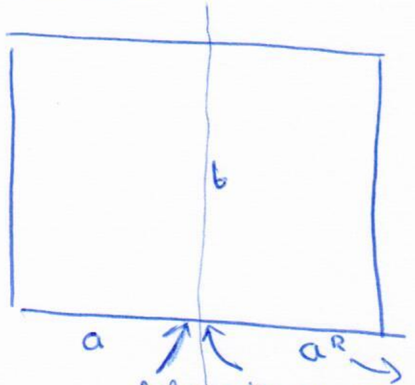
~~Conjec~~

How to prove Theorem 1?

$\mathbb{P}(\text{Flat}_n) \geq e^{-cn}$

↳ uniform probability on $Hom(B_n, H)$

Basic Idea:



$$\mathbb{P}(a\, a^R \mid b) = (\mathbb{P}(a \mid b))^2$$

$\cancel{\int_b \mathbb{P}(a a^R \mid b)}$ Choose a such that

$$\mathbb{P}(a) \geq \frac{1}{|H|^{4n}}$$

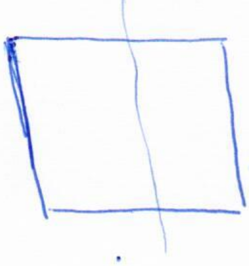
Then $\mathbb{P}(a\, a^R) = \int_b \mathbb{P}(a\, a^R \mid b) = \int_b \mathbb{P}(a \mid b)^2$

$$\geq \left(\int_b \mathbb{P}(a \mid b) \right)^2$$

$$= \mathbb{P}(a)^2$$

$$\geq \frac{1}{|H|^{8n}}.$$

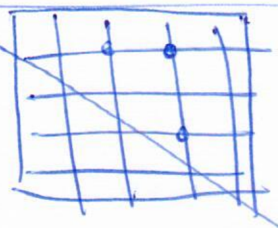
$\mathbb{P}$(periodic boundary condition) $\geq \frac{1}{|H|^{8n}}$.

⇝  ⇝ on a torus.

repeated use of same idea.

by repeatedly shifting the line of reflection.

Benjamini & Mossel '2000 $\Rightarrow$ Thm (1) for $X_{\substack{0 \\ 1-2}}$

B_n. 

~~Basic version:~~

Qn: $\mathbb{P}(X_i = v) > c$ (independent of n and F)

$X_F = v$

$F \cup \{i\}$ is connected

Connected set of even sites.

Qn: $v \sim w$ $\exists c$ s.t. $\forall\, i, F$, $\{i\}\cup\{F$ and F are connected sets. i is even.

$$\mathbb{P}\left(X_i = v \;\middle|\; X_j = \begin{cases} v & j\in F \text{ is } \cancel{\text{odd}} \text{ even} \\ w & j \in F \text{ is odd} \end{cases}\right) \geq c$$

~~Version.~~

Conjecture: $v \sim_H w$ $\exists c$ (independent of n ~~and F~~) s.t.

$\forall\, i, F \subset B_n$ ($\{i\}\cup F$ and F are connected)

$$\mathbb{P}\left(X_i = v \;\middle|\; X_j = \begin{cases} v & j\in F \text{ is even} \\ w & j \in F \text{ is odd} \end{cases}\right) \geq c.$$