Uniqueness of clusters in percolation

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For a ‘random’ x we want to understand whether there is an infinite cluster.

What is random?

Upper case X, Y are going to be random variables while lower case x, y are going to be deterministic.

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This is the Bernoulli site percolation model. Similar models can be considered where we colour edges instead of vertices (called bond percolation) and so on.

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This says that $E_{infinite}$ is independent of the values on a finite set, that is, given $x \in \{0,1\}^V$ we have that

$$\begin{aligned} \mathbb{P}_p(E_{infinite}, X(v) = x(v) \text{ for } v \in W) &= \mathbb{P}_p(E_{infinite}) \\ &\quad \mathbb{P}_p(X(v) = x(v) \text{ for } v \in W). \end{aligned}$$

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But then we get

$$\mathbb{P}_p(E_{infinite}) = \mathbb{P}_p(E_{infinite} \cap E_{infinite}) = (\mathbb{P}_p(E_{infinite}))^2$$

meaning that $\mathbb{P}_p(E_{infinite}) = 0$ or 1.

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By similar arguments as above we have that

$$\mathbb{P}_p(E_{infinitely\ infinite})$$

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Under some technical assumptions on G (like transitivity) it was proved by Häggstörm, Peres and Schonmann (1999) that there is a p_u such that for $p_c < p < p_u$, $\mu_p(E_{\text{infinitely infinite}}) = 1$ and for $p > p_u$, $\mu_p(E_{\text{infinitely infinite}}) = 0$.

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Now connect v_1 and v_2 by an edge to get a new graph G. Then in this new graph G, the number of infinite clusters is either 1 or 2, both with positive probability.

This is because the graph G is not very regular (and also not particularly interesting).

From here on by G we will mean the Cayley graph of a group (which will also denote the group).

In the case $\mathbb{Z}^d$ we will think of the group with the standard set of generators. Thus $\mathbb{Z}^2$ will mean the usual grid graph.

In general we can work with transitive and quasi-transitive graphs.

A little bit about the setting

The group G acts on the space of configurations $\{0,1\}^G$ in a natural way by shifts, namely, for all $g \in G$, $x \in \{0,1\}^G$ we have

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Previously we were interested in measures μ_p where for any finite set $A \subset G$ and $x \in \{0,1\}^G$ we had

$$\mu_p(x' : x'(g) = x(g) \text{ for } g \in A) = p^{\text{number of 1 in } x|_A}(1-p)^{\text{number of 0 in } x|_A}.$$

Finite energy

Let $p \leq 1/2$. An important property of μ_p is what is called “finite energy”, namely, given a finite set $A \subset G$, and $\phi : \{0,1\}^G \to \{0,1\}^G$ if

$$\phi(x) \text{ and } x \text{ have the same values on } G \setminus A$$

then for all sets of positive measure $E \subset \{0,1\}^G$ we have that

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A measure μ is said to have finite energy if there exists $c, C > 0$ such that

$$c\mu(E) \leq \mu(\phi(E)) \leq C\mu(E).$$

The main result

Theorem (Aizenman, Kesten and Newman, 1987)

Let μ *be a shift-invariant measure on* $\{0,1\}^{\mathbb{Z}^d}$ *with finite energy. Then*

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We will present the proof by Burton and Keane from two years later (also look at Häggström and Jonasson - 2006) which is less quantitative but at the same time can be generalised easily to other groups and settings.

Before diving into the proof let us study the hypothesis a bit.

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The latest progress was made by Hutchcroft (2020) who proved this for a large class of graphs (specifically transitive graphs whose automorphism group contains a non-unimodular subgroup).

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Without the finite energy assumption, for every $N \in \mathbb{N} \cup \{\infty\}$ Burton and Keane (1991) constructed an example with exactly N infinite clusters almost surely. The construction is a beautiful substitution scheme.

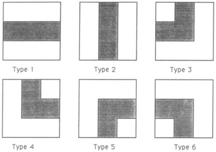

Fig. 1.

This construction is iterated. For example, a type 1 n-block is a $3 \cdot 5^n \times 3 \cdot 5^n$ configuration made up of 5^2 $(n-1)$-blocks whose types are arranged as in Figure 2.

5	6	5	1	6
2	2	2	5	3
3	2	2	2	5
5	3	2	2	2
4	1	1	4	3

Fig. 2.

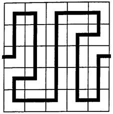

Fig. 3.

Proof

What do we want to prove?

A measure μ has finite energy if for all $\phi : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ which changes only finitely many coordinates, there are constants $c, C > 0$ such that

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It is enough to prove this for ergodic measures. For $n \in \mathbb{N} \cup \{\infty\} \cup \{0\}$ define

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We want to show that either $\mu(E_0) = 1$ or $\mu(E_1) = 1$.

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Suppose $\mu(E_i) = 1$ for some $i \geq 2$. Now note that

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Thus we have that either $\mu(E_0), \mu(E_1)$ or $\mu(E_\infty)$ is 1.

0, 1, ∞

A measure μ has finite energy if for all $\phi : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ which changes only finitely many coordinates, there are constants $c, C > 0$ such that
$$c\mu(E) < \mu(\phi(E)) < C\mu(E).$$

We have $\mu(E_i) = 0$ or 1 for all i. We want either $\mu(E_0) = 1$ or $\mu(E_1) = 1$.

Suppose $\mu(E_i) = 1$ for some $i \geq 2$. Now note that

$$E_i = \bigcup_{A \subset \mathbb{Z}^2 \text{ is finite}} \{\text{there are exactly } i \text{ infinite clusters touching } A' \text{ for all } A' \supset A\}.$$

Thus we have that there is some finite set $A \subset \mathbb{Z}^2$ such that

$$\mu(\{\text{there are exactly } i \text{ infinite clusters touching } A' \text{ for all } A' \supset A\}) > 0.$$

But then if $\phi_A : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}^{\mathbb{Z}^d}$ is a map which turns all the coordinates in A to be 1 we have by finite energy

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Thus we have that either $\mu(E_0), \mu(E_1)$ or $\mu(E_\infty)$ is 1. This generalises to all transitive graphs.

Estimating $\mu(E_\infty)$: We must have trifurcation points

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If for μ almost every x, there are infinitely many infinite clusters then there must be a large enough box B which intersects at least 3 infinite clusters.

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By changing the configuration on B, we can create a trifurcation point in B.

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By finite energy, μ almost every x has trifurcation points.

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For contradiction, it is enough to prove that there are no trifurcation points.

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For the sake of contradiction assume that $\mu(E_\infty) = 1$.

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If for μ almost every x, there are infinitely many infinite clusters then there must be a large enough box B which intersects at least 3 infinite clusters.

By changing the configuration on B, we can create a trifurcation point in B.

By finite energy, μ almost every x has trifurcation points.

For contradiction, it is enough to prove that there are no trifurcation points. This will come from the fact that the number of infinite clusters touching a box B is at most the size of the boundary of B.

Estimating $\mu(E_\infty)$: There must be outer-trifurcation points.

A vertex $v \in K$ is called a trifurcation point if removing v disconnects K into a disjoint union of three infinite clusters.

Let $K \subset \mathbb{Z}^d$ be an infinite connected set and $A \subset K$ be a finite set of trifurcation points.

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Let $K \subset \mathbb{Z}^d$ be an infinite connected set and $A \subset K$ be a finite set of trifurcation points. A trifurcation point $v \in A$ is called outer if $A \setminus \{v\}$ is contained in one of the three components of $K \setminus \{v\}$.

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Pick $v_1 \in A$. If v_1 is not outer then there must be v_2 and v_3 in A which belong to distinct connected components of $K \setminus \{v_1\}$.

If v_3 is not outer then there are at least two connected components of $K \setminus \{v_3\}$: one contains v_1, v_2 and another contains some v_4 in A.

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Continuing in this fashion we get a sequence $v_1, v_2, \ldots, v_n$ in A such that $v_1, v_2, \ldots, v_{n-1}$ is in one component of $K \setminus \{v_n\}$.

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This will eventually exhaust A and hence we find an outer trifurcation point.

Estimating $\mu(E_\infty)$: Removing trifurcation points gives a lot of connected components

Let $K \subset \mathbb{Z}^d$ be an infinite connected set. A vertex $v \in K$ is called a trifurcation point if removing v disconnects K into a disjoint union of three infinite clusters. Let $A \subset K$ be a finite set of trifurcation points. A trifurcation point $v \in A$ is called outer if $A \setminus \{v\}$ is contained in one of the three components of $K \setminus \{v\}$.

Claim: If A is a finite set of trifurcation points then $K \setminus A$ has at least $|A| + 2$ connected components.

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Let $v \in A'$ be outer.

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Let $v \in A'$ be outer. By the induction hypothesis, $K \setminus (A' \setminus \{v\})$ has at least $n + 2$ connected components.

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Let $v \in A'$ be outer. By the induction hypothesis, $K \setminus (A' \setminus \{v\})$ has at least $n + 2$ connected components. Since v is outer, $A' \setminus \{v\}$ is contained in exactly one connected component of $K \setminus \{v\}$.

Estimating $\mu(E_\infty)$: Removing trifurcation points gives a lot of connected components

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Let $v \in A'$ be outer. By the induction hypothesis, $K \setminus (A' \setminus \{v\})$ has at least $n + 2$ connected components. Since v is outer, $A' \setminus \{v\}$ is contained in exactly one connected component of $K \setminus \{v\}$. Thus removing v gives at least 1 more connected component. This completes the proof of the claim.

Proving $\mu(E_\infty) = 0$

Let $K \subset \mathbb{Z}^d$ be an infinite connected set. If A is a finite set of trifurcation points then $K \setminus A$ has at least $|A| + 2$ infinite clusters.

Proving $\mu(E_\infty) = 0$

Let $K \subset \mathbb{Z}^d$ be an infinite connected set. If A is a finite set of trifurcation points then $K \setminus A$ has at least $|A| + 2$ infinite clusters.

For the sake of contradiction we had assumed that $\mu(E_\infty) = 1$.

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Let $B := [-n, n]^d$ be a box in $\mathbb{Z}^d$. The number of infinite clusters of almost every x touching B is at most the boundary of B.

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For the sake of contradiction we had assumed that $\mu(E_\infty) = 1$.

Let $B := [-n, n]^d$ be a box in $\mathbb{Z}^d$. The number of infinite clusters of almost every x touching B is at most the boundary of B.

But by the ergodic theorem, a positive proportion of vertices in B must be trifurcation points. This leads to a contradiction and completes the proof.

A quantitative question: By Raphaël Cerf (heard from Gady Kozma)

Let μ be a ergodic measure on $\{0,1\}^{\mathbb{Z}^d}$ with finite energy.

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From Burton and Keane's result it can be extracted that the probability that there is a cluster in $[-n,n]^d$ touching the boundary which disconnects into at least three connected components (each touching the boundary) when the origin is removed is bounded (up to a constant) by $\frac{1}{n}$.

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Can this be generalised for more number of connected components?

Thank you!