Markov random fields and the Pivot property

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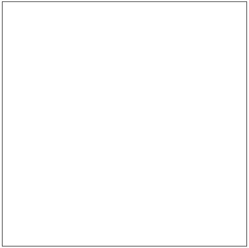
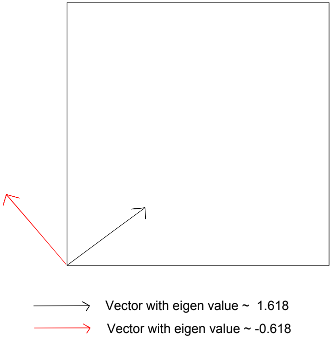
Outline

- Nearest neighbour shifts of finite type
- Topological Markov fields
- Markov random fields and Gibbs measures with nearest neighbour interactions
- Pivot property
- Examples: 3-coloured chessboard and the Square Island shift.

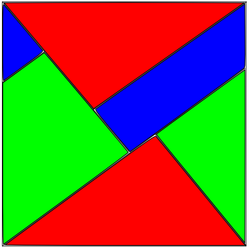
Consider a torus $\mathbb{R}^2/\mathbb{Z}^2$ with the map $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.



$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has two eigenvalues (~ 1.618 and $-.618$).



We can divide the torus into 3 parts by extending the eigendirections. These are called Markov partitions.



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This phenomenon is much more general: Any automorphism of the torus(with no eigenvalues of modulus 1) can be coded in a similar way.

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A nearest neighbour shift of finite type is a shift space such that $\mathcal{F}$ can be given by patterns on 'edges'.

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Figure : The 3-coloured chessboard in 2 dimensions

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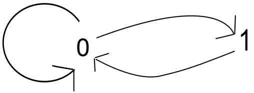

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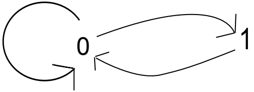

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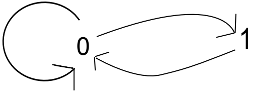

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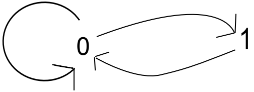

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For instance, the hard square model ($\mathcal{A} = \{0, 1\}$, $\mathcal{F} = \{11\}$) has a safe symbol viz. 0 but the 3-coloured chessboard ($\mathcal{A} = \{0, 1, 2\}$ and $\mathcal{F} = \{00, 11, 22\}$) does not have any safe symbol.

Topological Markov Fields

A topological Markov field is a shift space $X \subset \mathcal{A}^{\mathbb{Z}^d}$ with the 'conditional independence' property: for all finite subsets $F \subset \mathbb{Z}^d$, $x, y \in X$ satisfying $x = y$ on ∂F, $z \in \mathcal{A}^{\mathbb{Z}^d}$ given by

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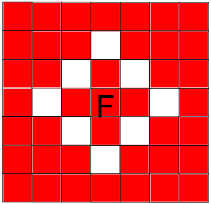
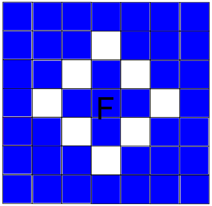
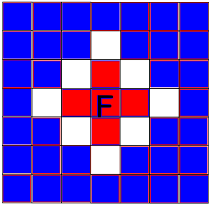


x



y



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Consider any one-dimensional shift space X. Make a two-dimensional shift space Y where the horizontal constraints come from X and the vertical direction is constant. If x and y agree on ∂F, they must agree on F. Therefore Y is a topological Markov field. There are uncountably many such shift spaces but there are only countably many nearest neighbour shift of finite type!!

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A Gibbs state with a nearest neighbor interaction V is a Markov random field μ such that for all $x \in supp(\mu)$ and $A, B \subset \mathbb{Z}^d$ finite satisfying $\partial A \subset B \subset A^c$

$$\mu([x]_A \,\Big|\, [x]_B) = \frac{\prod_{C \subset A \cup \partial A} e^{V([x]_C)}}{Z_{A, x|_{\partial A}}}$$

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Question: How can we weaken the hypothesis?

Pivot Property

A shift space X is said to satisfy the pivot property if for all $x, y \in X$ which differ only on finitely many sites there exists a chain $x = x^1, x^2, x^3, \ldots, x^n = y \in X$ such that x^i, x^{i+1} differ on at most a single site.

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Examples:

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- Domino tilings.

The 3-coloured Chessboard

The 3-coloured chessboard has the pivot property.

1	0	2	0	1	0	1
0	2	0	1	2	1	0
1	0	1	0	1	0	1
0	1	0	2	0	1	2
2	0	1	0	1	2	0
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2	0	1	2	1	2	0
0	2	0	1	0	1	2

The 3-coloured Chessboard

The 3-coloured chessboard has the pivot property.

1	0	2	0	1	0	1
0	2	0	1	2	1	0
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$$
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\frac{\mu([x]_F \mid [x]_{\partial F})}{\mu([y]_F \mid [x]_{\partial F})} &= \prod_{i=1}^{n-1} \frac{\mu([x^i]_F \mid [x^i]_{\partial F})}{\mu([x^{i+1}]_F \mid [x^i]_{\partial F})} \\
&= \prod_{i=1}^{n-1} \frac{\mu([x^i]_{m_i} \mid [x^i]_{\partial m_i})}{\mu([x^{i+1}]_{m_i} \mid [x^i]_{\partial m_i})}.
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Therefore the entire specification is determined by finitely many parameters viz. $\frac{\mu([x]_{0\cup\partial 0})}{\mu([y]_{0\cup\partial 0})}$ for configurations x, y which differ only at 0, the origin.

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Theorem

Given a shift space with the pivot property the space of specifications on that shift space can be parametrised by finitely many parameters.

Question: Suppose we are given a nearest neighbour shift of finite type with the pivot property. Is there an algorithm to determine the number of parameters which describes the specification?

Thus a specification supported on the 3-coloured chessboard is determined the quantities $v_1 = \dfrac{\mu(\begin{bmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{bmatrix})}{\mu(\begin{bmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{bmatrix})}$, Thus a specification supported on the 3-coloured chessboard is

determined the quantities $v_1 = \dfrac{\mu(\left[\begin{smallmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{smallmatrix}\right])}{\mu(\left[\begin{smallmatrix} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{smallmatrix}\right])}$, $v_2 = \dfrac{\mu(\left[\begin{smallmatrix} & 2 & \\ 2 & 1 & 2 \\ & 2 & \end{smallmatrix}\right])}{\mu(\left[\begin{smallmatrix} & 2 & \\ 2 & 0 & 2 \\ & 2 & \end{smallmatrix}\right])}$

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$$\begin{aligned} v_1 &= exp(V(01) + V(10) + V(\substack{0\\1}) + V(\substack{0\\1}) \\ &\quad -V(21) - V(12) - V(\substack{2\\1}) - V(\substack{1\\2})), \\ v_2 &= exp(V(12) + V(21) + V(\substack{2\\1}) + V(\substack{1\\2}) \\ &\quad -V(02) - V(20) - V(\substack{0\\2}) - V(\substack{2\\0})), \\ v_3 &= exp(V(02) + V(20) + V(\substack{2\\0}) + V(\substack{0\\2}) \\ &\quad -V(01) - V(10) - V(\substack{0\\1}) - V(\substack{1\\0})). \end{aligned}$$

Thus μ is Gibbs if and only if $v_1 v_2 v_3 = 1$.

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What if the pivot property does not hold?

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Thus the Hammersley-Clifford type conclusion holds for fully supported measures.

What if the pivot property does not hold? Every 1 dimensional nearest neighbour shift of finite type has the generalised pivot property.

Square Island Shift

Inspiration from checkerboard island shift by Quas and Şahin.

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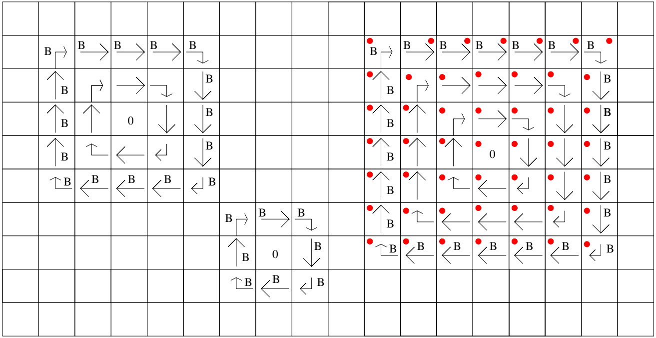

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There are two kinds of squares: ones with red dots and ones without red dots which float in a sea of blanks.

The Square Island shift does not have the generalised pivot property.

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There is no way to switch from a big square with red dots to a big square without red dots making single site changes(or even bigger regional changes).

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Question: Can more uniform mixing conditions help?

Thank You!