CHAPTER 12 REVIEW EXERCISES

1. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.
   a. The equation $4x - 3y = 12$ describes a line in $\mathbf{R}^3$.
   b. The equation $z^2 = 2x^2 - 6y^2$ determines z as a single function of x and y.
   c. If f has continuous partial derivatives of all orders, then $f_{xxy} = f_{yyx}$.
   d. Given the surface $z = f(x, y)$, the gradient $\nabla f(a, b)$ lies in the plane tangent to the surface at $(a, b, f(a, b))$.
   e. There is always a plane orthogonal to both of two distinct intersecting planes.

2. **Equations of planes** Consider the plane that passes through the point $(6, 0, 1)$ with a normal vector $\mathbf{n} = \langle 3, 4, -6\rangle$.
   a. Find an equation of the plane.
   b. Find the intercepts of the plane with the three coordinate axes.
   c. Make a sketch of the plane.

3. **Equations of planes** Consider the plane passing through the points $(0, 0, 3)$, $(1, 0, -6)$, and $(1, 2, 3)$.
   a. Find an equation of the plane.
   b. Find the intercepts of the plane with the three coordinate axes.
   c. Make a sketch of the plane.

4–5. Intersecting planes *Find an equation of the line that forms the intersection of the following planes Q and R.*

4. Q: $2x + y - z = 0$, R: $-x + y + z = 1$

5. Q: $-3x + y + 2z = 0$, R: $3x + 3y + 4z - 12 = 0$

6–7. Equations of planes *Find an equation of the following planes.*

6. The plane passing through $(2, -3, 1)$ normal to the line $\langle x, y, z\rangle = \langle 2 + t, 3t, 2 - 3t\rangle$

7. The plane passing through $(-2, 3, 1)$, $(1, 1, 0)$, and $(-1, 0, 1)$

8–22. Identifying surfaces *Consider the surfaces defined by the following equations.*
   a. Identify and briefly describe the surface.
   b. Find the xy-, xz-, and yz-traces, if they exist.
   c. Find the intercepts with the three coordinate axes, if they exist.
   d. Make a sketch of the surface.

8. $z - \sqrt{x} = 0$

9. $3z = \dfrac{x^2}{12} - \dfrac{y^2}{48}$

10. $\dfrac{x^2}{100} + 4y^2 + \dfrac{z^2}{16} = 1$

11. $y^2 = 4x^2 + z^2/25$

12. $\dfrac{4x^2}{9} + \dfrac{9z^2}{4} = y^2$

13. $4z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$

14. $\dfrac{x^2}{16} + \dfrac{z^2}{36} - \dfrac{y^2}{100} = 1$

15. $y^2 + 4z^2 - 2x^2 = 1$

16. $-\dfrac{x^2}{16} + \dfrac{z^2}{36} - \dfrac{y^2}{25} = 4$

17. $\dfrac{x^2}{4} + \dfrac{y^2}{16} - z^2 = 4$

18. $x = \dfrac{y^2}{64} - \dfrac{z^2}{9}$

19. $\dfrac{x^2}{4} + \dfrac{y^2}{16} + z^2 = 4$

20. $y - e^{-x} = 0$

21. $\dfrac{y^2}{49} + \dfrac{x^2}{9} = \dfrac{z^2}{64}$

22. $y = 4x^2 + \dfrac{z^2}{9}$

23–26. Domains *Find the domain of the following functions. Make a rough sketch of the domain in the xy-plane.*

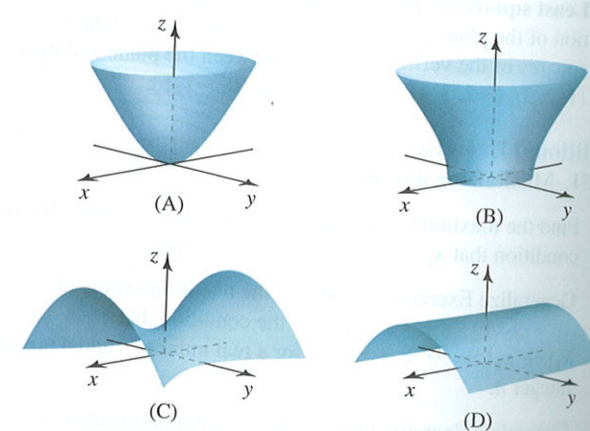
23. $f(x, y) = \dfrac{1}{y^2 + x^2}$

24. $f(x, y) = \ln(xy)$

25. $f(x, y) = \sqrt{x - y^2}$

26. $f(x, y) = \tan(x + y)$

27. **Matching surfaces** Match functions a–d with surfaces A–D in the figure.
   a. $z = \sqrt{2x^2 + 3y^2 + 1} - 1$
   b. $z = -3y^2$
   c. $z = 2x^2 - 3y^2 + 1$
   d. $z = \sqrt{2x^2 + 3y^2 - 1}$

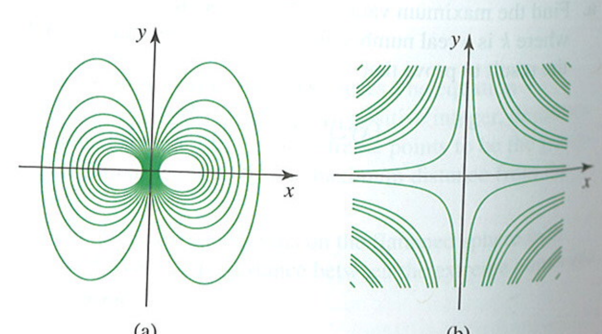


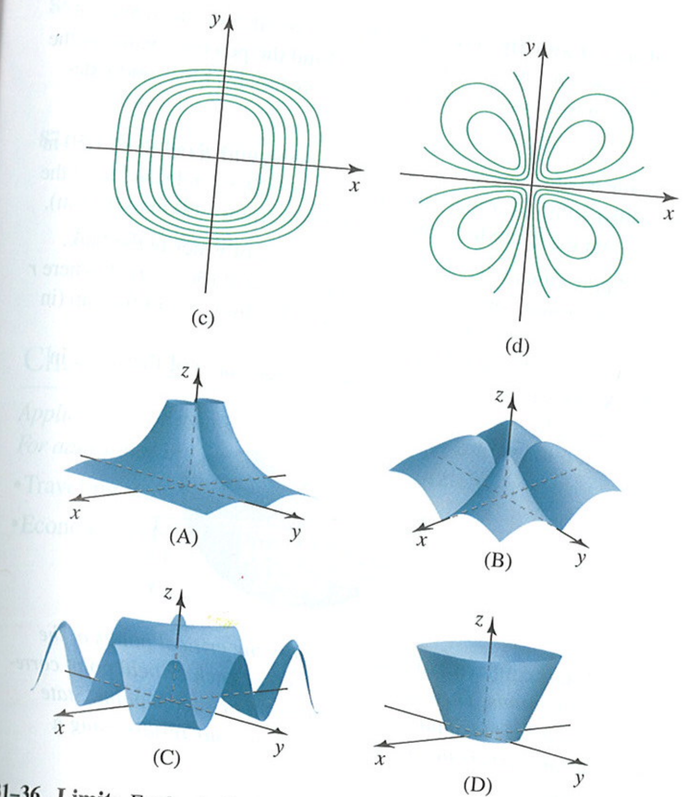
28–29. Level curves *Make a sketch of several level curves of the following functions. Label at least two level curves with their z-values.*

28. $f(x, y) = x^2 - y$

29. $f(x, y) = 2x^2 + 4y^2$

30. **Matching level curves with surfaces** Match level curve plots a–d with surfaces A–D.





31–36. Limits *Evaluate the following limits or determine that they do not exist.*

31. $\displaystyle\lim_{(x,y)\to(4,-2)} (10x - 5y + 6xy)$

32. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{xy}{x + y}$

33. $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x^2 - y^2}{x^2 - xy - 2y^2}$

34. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2 y}{x^4 + 2y^2}$

35. $\displaystyle\lim_{(x,y,z)\to(\frac{\pi}{2},0,\frac{\pi}{2})} 4\cos y \sin\sqrt{xz}$

36. $\displaystyle\lim_{(x,y,z)\to(5,2,-3)} \tan^{-1}\left(\frac{x + y^2}{z^2}\right)$

37–40. Partial derivatives *Find the first partial derivatives of the following functions.*

37. $f(x, y) = xy\, e^{xy}$

38. $g(u, v) = u\cos v - v\sin u$

39. $f(x, y, z) = e^{x+2y+3z}$

40. $H(p, q, r) = p^2\sqrt{q + r}$

41–42. Laplace's equation *Verify that the following functions satisfy Laplace's equation,* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

41. $u(x, y) = y(3x^2 - y^2)$

42. $u(x, y) = \ln(x^2 + y^2)$

43. **Region between spheres** Two spheres have the same center with radii r and R, where $0 < r < R$. The volume of the region between the spheres is $V(r, R) = \dfrac{4\pi}{3}(R^3 - r^3)$.
   a. First, use your intuition. If r is held fixed, how does V change as R increases? What is the sign of V_R? If R is held fixed, how does V change as r increases (up to the value of R)? What is the sign of V_r?

   b. Compute V_r and V_R. Are the results consistent with part (a)?
   c. Consider spheres with $R = 3$ and $r = 1$. Does the volume change more if R is increased by $\Delta R = 0.1$ (with r fixed) or if r is decreased by $\Delta r = 0.1$ (with R fixed)?

44–47. Chain Rule *Use the Chain Rule to evaluate the following derivatives.*

44. $w'(t)$, where $w = xy\sin z$, $x = t^2$, $y = 4t^3$, and $z = t + 1$

45. $w'(t)$, where $w = \sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, and $z = \cos t$

46. w_s and w_t, where $w = xyz$, $x = 2st$, $y = st^2$, and $z = s^2 t$

47. w_r, w_s *and* w_t, where $w = \ln(x^2 + y^2 + 1)$, $x = rst$, and $y = r + s + t$

48–49. Implicit differentiation *Find dy/dx for the following implicit relations.*

48. $2x^2 + 3xy - 3y^4 = 2$

49. $y\ln(x^2 + y^2) = 4$

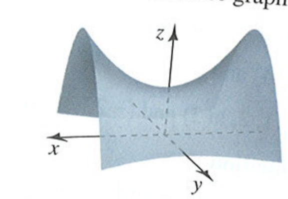
50–51. Walking on a surface *Consider the following surfaces and parameterized curves C in the xy-plane.*
   a. In each case find $z'(t)$ on C.
   b. Imagine that you are walking on the surface directly above C. Find the values of t for which you are walking uphill.

50. $z = 4x^2 + y^2 - 2$; C: $x = \cos t$, $y = \sin t$, for $0 \le t \le 2\pi$

51. $z = x^2 - 2y^2 + 4$; C: $x = 2\cos t$, $y = 2\sin t$, for $0 \le t \le 2\pi$

52. **Constant volume cones** Suppose the radius of a right circular cone increases as $r(t) = t^a$ and the height decreases as $h(t) = t^{-b}$, for $t \ge 1$, where a and b are positive constants. What is the relationship between a and b such that the volume of the cone remains constant (that is, $V'(t) = 0$, where $V = (\pi/3)r^2 h$)?

53. **Directional derivatives** Consider the function $f(x, y) = 2x^2 - 4y^2 + 10$, whose graph is shown in the figure.



   a. Fill in the table showing the value of the directional derivative at points (a, b) in the direction θ.

	$(a, b) = (0, 0)$	$(a, b) = (2, 0)$	$(a, b) = (1, 1)$
$\theta = \pi/4$			
$\theta = 3\pi/4$			
$\theta = 5\pi/4$			

   b. Indicate in a sketch of the xy-plane the point and direction for each of the table entries in part (a).

54–57. Computing gradients *Compute the gradient of the following functions, evaluate it at the given point, and evaluate the directional derivative in the given direction.*

54. $f(x, y) = \sin(x - 2y)$; $(-1, -5)$; $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$

55. $h(x, y) = \sqrt{2 + x^2 + 2y^2}$; $(2, 1)$; $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

56. $f(x, y, z) = xy + yz + xz + 4$; $(2, -2, 1)$; $\mathbf{u} = \langle 0, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$

57. $f(x, y, z) = 1 + \sin(x + 2y - z)$; $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{-\pi}{6}\right)$; $\mathbf{u} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

58–59. Direction of steepest ascent and descent

a. *Find the unit vectors that give the direction of steepest ascent and steepest descent at P.*

b. *Find a unit vector that points in a direction of no change.*

58. $f(x, y) = \ln(1 + xy)$; $P(2, 3)$

59. $f(x, y) = \sqrt{4 - x^2 - y^2}$; $P(-1, 1)$

60–61. Level curves *Consider the paraboloid* $f(x, y) = 8 - 2x^2 - y^2$. *For the following level curves* $f(x, y) = C$ *and points* (a, b), *compute the slope of the line tangent to the level curve at* (a, b) *and verify that the tangent line is orthogonal to the gradient at that point.*

60. $f(x, y) = 5$; $(a, b) = (1, 1)$

61. $f(x, y) = 0$; $(a, b) = (2, 0)$

62. Directions of zero change Find the directions in which the function $f(x, y) = 4x^2 - y^2$ has zero change at the point $(1, 1, 3)$. Express the directions in terms of unit vectors.

63. Electric potential due to a charged cylinder. An infinitely long charged cylinder of radius R with its axis along the z-axis has an electric potential $V = k \ln(R/r)$, where r is the distance between a variable point $P(x, y)$ and the axis of the cylinder ($r^2 = x^2 + y^2$) and k is a physical constant. The electric field at a point (x, y) in the xy-plane is given by $\mathbf{E} = -\nabla V$, where ∇V is the two-dimensional gradient. Compute the electric field at a point (x, y) with $r > R$.

64–67. Tangent planes *Find an equation of the plane tangent to the following surfaces at the given points.*

64. $xy \sin z - 1 = 0$; $\left(1, 2, \frac{\pi}{6}\right)$ and $\left(-2, -1, \frac{5\pi}{6}\right)$

65. $yze^{xz} - 8 = 0$; $(0, 2, 4)$ and $(0, -8, -1)$

66. $z = x^2 e^{x-y}$; $(2, 2, 4)$ and $(-1, -1, 1)$

67. $z = \ln(1 + xy)$; $(1, 2, \ln 3)$ and $(-2, -1, \ln 3)$

T **68–69. Linear approximation**

a. *Find the linear approximation (the equation of the tangent plane) at the point* (a, b).

b. *Use part (a) to estimate the given function value.*

68. $f(x, y) = 4 \cos(2x - y)$; $(a, b) = \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$; estimate $f(0.8, 0.8)$.

69. $f(x, y) = (x + y)e^{xy}$; $(a, b) = (2, 0)$; estimate $f(1.95, 0.05)$.

70. Changes in a function Estimate the change in the function $f(x, y) = -2y^2 + 3x^2 + xy$ when (x, y) changes from $(1, -2)$ to $(1.05, -1.9)$.

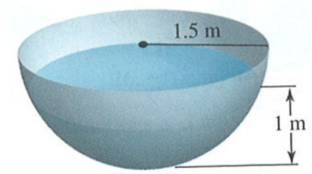
71. Volume of a cylinder The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Find the approximate percent change in the volume when the radius decreases by 3% and the height increases by 2%.

72. Volume of an ellipsoid The volume of an ellipsoid with axes of length $2a$, $2b$, and $2c$ is $V = \pi abc$. Find the percent change in the volume when a increases by 2%, b increases by 1.5%, and c decreases by 2.5%.

73. Water level changes A hemispherical tank with a radius of 1.50 m is filled with water to a depth of 1.00 m. Water is released from the tank and the water level drops by 0.05 m (from 1.00 m to 0.95 m).

a. Approximate the change in the volume of water in the tank. The volume of a spherical cap is $V = \pi h^2 (3r - h)/3$, where r is the radius of the sphere and h is the thickness of the cap (in this case, the depth of the water).

b. Approximate the change in the surface area of the water in the tank.



74–77. Analyzing critical points *Identify the critical points of the following functions. Then determine whether each critical point corresponds to a local maximum, local minimum, or saddle point. State when your analysis is inconclusive. Confirm your results using a graphing utility.*

74. $f(x, y) = x^4 + y^4 - 16xy$

75. $f(x, y) = x^3/3 - y^3/3 + 2xy$

76. $f(x, y) = xy(2 + x)(y - 3)$

77. $f(x, y) = 10 - x^3 - y^3 - 3x^2 + 3y^2$

78–79. Absolute maxima and minima *Find the absolute maximum and minimum values of the following functions on the specified set.*

78. $f(x, y) = x^3/3 - y^3/3 + 2xy$ on the rectangle $\{(x, y): 0 \le x \le 3, -1 \le y \le 1\}$

79. $f(x, y) = x^4 + y^4 - 4xy + 1$ on the square $\{(x, y): -2 \le x \le 2, -2 \le y \le 2\}$

80. Least distance What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing you have found an absolute minimum of the distance function.

81–84. Lagrange multipliers *Use Lagrange multipliers to find the minimum and maximum values of f subject to the given constraint.*

81. $f(x, y) = x + 2y$ subject to $x^4 + y^4 = 1$

82. $f(x, y) = x^2y^2$ subject to $2x^2 + y^2 = 1$

83. $f(x, y, z) = x + 2y - z$ subject to $x^2 + y^2 + z^2 = 1$

84. $f(x, y, z) = x^2y^2z$ subject to $2x^2 + y^2 + z^2 = 25$

85. Maximum perimeter rectangle Use Lagrange multipliers to find the dimensions of the rectangle with the maximum perimeter that can be inscribed with sides parallel to the coordinate axes in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

86. Minimum surface area cylinder Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of 32π in^3.

87. Minimum distance to a cone Find the point(s) on the cone $z^2 - x^2 - y^2 = 0$ that are closest to the point $(1, 3, 1)$. Give an argument showing you have found an absolute minimum of the distance function.

88. Gradient of a distance function Let $P_0(a, b, c)$ be a fixed point in $\mathbf{R}^3$ and let $d(x, y, z)$ be the distance between P_0 and a variable point $P(x, y, z)$.

a. Compute $\nabla d(x, y, z)$.

b. Show that $\nabla d(x, y, z)$ points in the direction from P_0 to P and has magnitude 1 for all (x, y, z).

c. Describe the level surfaces of d and give the direction of $\nabla d(x, y, z)$ relative to the level surfaces of d.

d. Discuss $\lim_{P \to P_0} \nabla d(x, y, z)$.

Chapter 12 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- Traveling waves
- Economic production functions
- Ecological diversity