Trignometric Integrals

11th February

What have we leamt -

FTOC - Area functions as antiderivatives.

- Derivatives of integrals.

- Substitution Method

- Integration by parts.

$$\int_{1}^{4} \frac{\sin\left(\frac{\pi}{x}\right)}{x^2}\,dx.$$

$u = \frac{\pi}{x}$

$du = \frac{d\left(\frac{\pi}{x}\right)}{dx}\,dx = -\frac{\pi}{x^2}\,dx$

$\Rightarrow -\frac{1}{\pi}\,du = \frac{1}{x^2}\,dx.$

$x = 1 \Rightarrow u = \frac{\pi}{1} = \pi$

$x = 4 \Rightarrow u = \frac{\pi}{4}.$

Substituting we get

$$\int_{\pi}^{\pi/4} \sin(u)\left(-\frac{1}{\pi}\,du\right)$$

$$= -\frac{1}{\pi}\int_{\pi}^{\pi/4} \sin(u)\,du$$

$$= -\frac{1}{\pi}\left(-\cos(u)\right)\Big|_{\pi}^{\pi/4} = -\frac{1}{\pi}\left\{\left(-\cos\frac{\pi}{4}\right) - \left(-\cos(\pi)\right)\right\}$$

$$\underset{\text{Check}}{=} \frac{1}{\pi}\left(\frac{1}{\sqrt{2}} + 1\right)$$

Integration by Parts (IBP).

$$\int u\,dv = uv - \int v\,du.$$

Order for choosing u

L og
I nverse (arctanx, arcsinx)
A lgebraic
T rignometric
E xponential.
D.

For definite integrals

$$\int_a^b u(x)\,v'(x)\,dx = u(x)\,v(x)\Big|_a^b - \int_a^b v(x)\,u'(x)\,dx.$$

$\int e^x \sin x\,dx.$

- Substituting

$= \int u\,dv$

$= uv - \int v\,du$ (IBP)

$= \sin(x)e^x - \int e^x \cos x\,dx.$ ----- (1)

$u = \sin x \quad dv = e^x dx$

$du = \frac{d}{dx}(\sin x)\,dx \quad v = \int e^x dx = e^x$

$= \cos x\,dx$

$\int e^x \cos x\,dx$

$= \int u\,dv$

$= uv - \int v\,du$ (IBP)

$= \cos(x)\,e^x - \int e^x(-\sin x)\,dx$

$= \cos(x)e^x + \int e^x \sin x\,dx.$

$u = \cos x \quad dv = e^x dx$

$du = \frac{d}{dx}(\cos x)\,dx \quad v = \int e^x dx = e^x$

$\cancel{= \cos x\,dx}$

$= -\sin x\,dx$

Substituting in (1), we get.

$$\int e^x \sin x \, dx = \sin x \, e^x - \cos(x)\, e^x - \int e^x \sin x \, dx + C$$

$$\Rightarrow 2\int e^x \sin x \, dx = e^x(\sin x - \cos(x)) + C$$

$$\Rightarrow \int e^x \sin x \, dx = \tfrac{1}{2} e^x(\sin x - \cos(x)) + C.$$

Combination: $\int_1^5 \sin(\ln(x))\, dx.$

$u = \ln(x) \Longleftrightarrow e^u = x$

$du = \frac{d}{dx}(\ln(x))\, dx = \frac{1}{x} dx$

$\Rightarrow dx = x\, du = e^u du.$

$x = 1 \Rightarrow u = 0$

$x = 5 \Rightarrow u = \ln 5$

Substituting, we get,

$$\int_0^{\ln 5} \sin(u)\, e^u \, du.$$

– Now use previous problem.

~~Answer $\frac{1}{2}$ sin(ln(5)) – cos(ln~~

Answer: $\frac{5}{2}\left(\sin(\ln(5))\right) - \frac{5}{2}\left(\cos(\ln(5))\right) + \frac{1}{2}$

FBR: Key to substitution: find u getting rid of which you know how to integrate.

$$\int_1^2 \frac{(\ln x)^2}{x^2}\, dx.$$

$\ln x = u \iff x = e^u$

$dx = \frac{d}{du}(e^u)\, du = e^u du$

$x=1 \Rightarrow u=0$

$x=2 \Rightarrow u=\ln 2.$

Substituting, we get

$$\int_0^{\ln 2} \frac{u^2}{(e^u)^2} e^u du$$

$$= \int_0^{\ln 2} u^2 e^{-u} du$$

$$= \int_0^{\ln 2} x^2 e^{-x} dx$$

Integrate without limits.

$$\int x^2 e^{-x} dx.$$

$u = x^2$ $\quad dv = e^{-x} dx.$

$du = \frac{d}{dx}(x^2) dx$ $\quad v = \int e^{-x} dx$

$= 2x\, dx$ $\quad = \frac{e^{-x}}{-1} = -e^{-x}$

Substituting

$= \int u\, dv.$

$= uv - \int v\, du$ [IBP]

$= x^2(-e^{-x}) - \int (-e^{-x}) 2x\, dx$

$= -x^2 e^{-x} + 2\int x e^{-x} dx.$

Now use IBP again, ~~for~~ to get

__Check__: $\int x e^{-x} dx = -x e^{-x} - e^{-x} + C.$

Substituting, we get,

__Check:__ $\int x^2 e^{-x} dx = -e^{x}(x^2+2x+2) + C$

Put limits back to get

$$\int_0^{\ln 2} x^2 e^{-x}\,dx = -e^{-x}(x^2+2x+2)\Big|_0^{\ln 2}$$

$\boxed{\text{Check}}$ $= 2 - \frac{1}{2}\left((\ln(2))^2 + 2\ln(2) + 2\right)$

Trigonometric Integrals

$$\int \sin^m(x)\cos^n(x)\,dx.$$

1) If m is odd, rewrite $\sin^{m-1}(x)$ in terms of $\cos(x)$ and use $u = \cos(x)$ substitution.

$(du = -\sin x\,dx)$

$$\int \sin^m(x)\cos^n(x)\,dx = -\int \underbrace{\sin^{m-1}(x)}\cdot\cos^n(x)\,(\underbrace{-\sin(x)dx}_{du})$$

Change into $\cos(x)$ by $\sin^2(x) = 1-\cos^2 x$.

2) If n is odd, rewrite $\cos^{n-1}(x)$ is terms of $\sin(x)$ and use $u = \sin x$ substitution.

$(du = \cos x\,dx)$

$$\int \sin^m(x)\cos^n(x)\,dx = \int \sin^m(x)\,\underbrace{\cos^{n-1}(x)}\,\underbrace{\cos(x)\,dx}_{du}.$$

Change into $\sin(x)$ by $\cos^2(x) = 1-\sin^2(x)$.

3) If m, n are even, Use half angle formulaes repeatedly to convert into type 1 and 2

$\cdot \sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\quad \cdot \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\cdot \sin 2\theta = 2\sin\theta\cos\theta$

$\cdot \sin(x+y)$, $\sin(x-y)$, $\cos(x+y)$, $\cos(x-y)$

$\int \sin^7(x) \cos^2 x \, dx$

$\cancel{= \int (\sin^2 x)^3}$

$= \int \sin^6(x) \cos^2(x) \sin(x) \, dx$

$\cancel{= \int (1-\sin} \; = \int (\sin^2 x)^3 \cos^2(x) \sin x \, dx$

$= \int (1-\cos^2(x))^3 \cdot \cos^2(x) \sin x \, dx$

$u = \cos x$

$du = \frac{d}{dx}(\cos x) dx$

$= -\sin x \, dx$

Substituting, we get

$\int (1-u^2)^3 \cdot u^2 (-du) = -\int (1-u^2)^3 u^2 du$

Check

$= -\frac{u^3}{3} + \frac{3u^5}{5} - \frac{3u^7}{7} + \frac{u^9}{9} + C$

Substituting back $u = \cos x$, we get,

$-\frac{(\cos(x))^3}{3} - 3\frac{(\cos(x))^5}{5} + \frac{3}{7}(\cos x)^7 - \frac{1}{9}(\cos x)^9 + C$

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{4} \, 4 \sin^2 x \cos^2 x \, dx$$

$$= \frac{1}{4} \int (2 \sin x \cos x)^2 \, dx$$

$$= \frac{1}{4} \int \sin^2 2x \, dx$$

$$= \frac{1}{8} \int 2 \sin^2 2x \, dx$$

$$= \frac{1}{8} \int \big(1 - \cos(4x)\big) \, dx$$

$$= \frac{1}{8} \int dx - \frac{1}{8} \int \cos(4x) \, dx$$

$$= \frac{1}{8} x - \frac{1}{8} \left(\frac{\sin 4x}{4} \right) + C$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

$$\int \tan^m(x) \sec^n(x) \, dx.$$

1) <u>n even</u> (n=0 as well)

Rewrite $\cancel{\tan^{n-2}(x)}$ $\sec^{n-2}(x)$ in terms of $\cancel{\tan(x)}$ $\tan(x)$ and use $u = \tan x$ ($du = \sec^2 x \, dx$)

$$\int \tan^m(x) \sec^n(x) \, dx = \int \tan^m(x) \underbrace{\sec^{n-2}(x)}_{\text{Rewrite in terms of } \tan(x) \text{ using } \sec^2(x) = \tan^2(x)+1.} \underbrace{\sec^2(x) \, dx}_{du}$$

Rewrite in terms of $\tan(x)$ using $\sec^2(x) = \tan^2(x)+1$.

2) <u>m odd</u>. Rewrite $\tan^{m-1}(x)$ in terms of $\sec(x)$ and use $u = \sec(x)$
($du = \sec(x)\tan(x)\,dx$)

$$\int \tan^m(x)\sec^n(x)\,dx = \int \underbrace{\tan^{m-1}(x)}_{\text{rewrite in terms of } \sec(x) \text{ using } \tan^2(x) = \sec^2(x) - 1}\sec^{n-1}(x)\underbrace{\sec(x)\tan(x)\,dx}_{du}$$

3) n odd, m even. (Pray you do not see it).

Rewrite $\tan^m(x)$ in terms of $\sec(x)$ to convert to a type $\int \sec^n(x)\,dx$.

Use integration by parts with

~~$u = \sec^{n-2}(x)$~~

$u = \sec^{n-2}(x) \quad dv = \sec^2(x)\,dx$

and <u>repeat</u> till you get down to $\sec(x)$ / $\sec^2 x$.

Use similar steps for
$\int \cot^m(x)\csc^n(x)\,dx$