PROBABILITY

11th March

Can be f, Y ..
Letter does not matter.

Random Variable. is a function X : (Situations) $\longrightarrow$ Value

For instance. X could be the marks obtained by a person.

X(exam results) = marks obtained.

X could be the time of arrival of a bus.

X(today) = time the bus arrived.

we are interested in the probability of these events.

Random Variable

discrete. — Continuous.

Takes discrete value, roll of dice, tossing of coin.

Takes continuous values like arrival time of a bus, life time of a ~~bulb~~ bulb.

Study this.

Probability ~~deniss~~ density function → density of probability
(p d f)

~~It is a~~

~~A func~~ A random variable X has (pdf) $\cancel{f}$

$\cancel{f : \mathbb{P}}$ f is

$$\mathbb{P}r\,(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

Probability that X takes values between a and b.

For continuous random variable.

$$\cancel{\mathbb{P}r}\, \Pr\,(a \leq X \leq b) = \mathbb{P}(a < X \leq b)$$
$$= \mathbb{P}(a \leq X < b)$$
$$= \mathbb{P}(a < X < b).$$

Thus f is attached with some random variable.

Properties → Defines what a pdf is.

1) $f(x) \geq 0$ [Density should be non-negative]

2) $\int_{-\infty}^{\infty} f(t)\,dt = 1$ [Total probability 1]

Example: 1) Suppose $f(x) = kxe^{-kx}$ for $-\infty < x < \infty$

Can it be a pdf.?

Ans: If $k > 0$. $f(-1) = -ke^{k} < 0$.
(contradicts (1))

If $k < 0$ $f(1) = ke^{-k} < 0$
(contradicts (1) again)

Thus f cannot be a pdf.

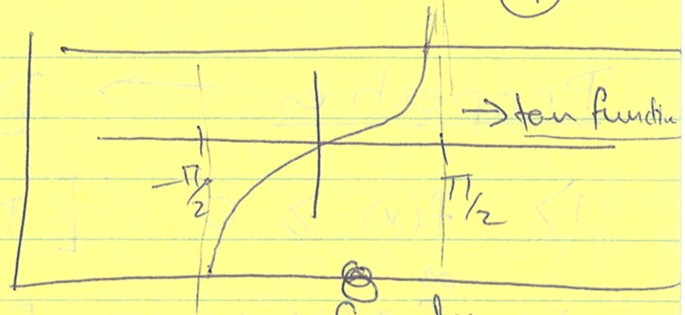
2) Find k such that ~~$f(x) = \frac{k}{1+x^2}$~~ $f(x) = \frac{k}{1+x^2}$ is a pdf.

Solution: By property (2)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Improper integral on unbounded interval.

$$\int_{-\infty}^{\infty} \frac{k\,dx}{(1+x^2)} = k\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$



$$= k\lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} + k\lim_{b\to-\infty}\int_b^0 \frac{dx}{1+x^2}$$

$$\int_0^b \frac{dx}{1+x^2} = $$

$$= \int_0^{\arctan(b)} \frac{\cancel{\sec^2 u}\,du}{\cancel{1+\tan^2 u} = \sec^2 u}$$

$x = \tan u$

$dx = \sec^2 u\,du$

$x = 0 \quad u = \arctan(0) = 0$

$x = b \quad u = \arctan(b)$

$$= \int_0^{\arctan(b)} \frac{\cancel{\sec^2 u}\,du}{\cancel{\sec^2 u}} = \int_0^{\arctan(b)} du = \arctan(b) - 0$$
$$= \arctan(b)$$

$$\lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\arctan(b) = \frac{\pi}{2}$$

$$\int_{b}^{0} \frac{dx}{1+x^2} = -\int_0^b \frac{dx}{1+x^2} = -\arctan(b)$$

$$\lim_{b\to-\infty}\int_b^0 \frac{dx}{1+x^2} = \lim_{b\to-\infty} -\arctan(b) = -\left(-\frac{\pi}{2}\right)$$
$$= \frac{\pi}{2}$$

$$\therefore \int_{-\infty}^{\infty}\frac{k\,dx}{1+x^2} = k\frac{\pi}{2} + k\left(\frac{\pi}{2}\right) = k\pi$$

But $\int_{-\infty}^{\infty} \frac{k\, dx}{1+x^2} = 1$ → pdf.

$\therefore \; k\pi = 1$

$\Rightarrow \; k = \frac{1}{\pi}$ (Answer).

Cumulative density function (cdf)

Cumulative density function for a random variable X is

$$F(x) = \mathbb{P}r(X \leq x)$$

$F(x)$ → cdf.

$X \leq x$ → Probability X takes value $\leq x$.

Relation between cdf and pdf.

Suppose cdf is $F(x)$ and pdf is $f(x)$

Then. (1) $\int_{-\infty}^{x} f(x)\, dx = \mathbb{P}(X \leq x) = F(x)$.

(2) $F'(x) = f(x)$.

(1) How to obtain cdf given pdf?

(2) How to obtain pdf given cdf?

Properties → Defines what a cdf is.

1> $\boxed{0 \leq F(x) \leq 1}$ [Probabilities are between 0 and 1]

2> $\boxed{\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} F(x) = 1.}$

$$\lim_{x \to -\infty} \mathbb{P}(X \leq x) \qquad\qquad \lim_{x \to \infty} \mathbb{P}(X \leq x) = \mathbb{P}(X < \infty)$$

(Probability on nothing happening. Probability of everything.

3> $\boxed{F \text{ is non decreasing.}}$

$$F(1) \leq F(2)$$

$$\mathbb{P}(X \leq 1) \leq \mathbb{P}(X \leq 2).$$

Can be any two numbers. $a \leq b$ ~~then~~.

then. $F(a) \leq F(b)$.

Examples.

~~Given pdf f(x)~~

Let $f(x) = \cancel{\frac{k}{1+x^2}}$ $\cancel{ke^{-x}}$ $k|x|e^{-x^2}$

Find k such that f is a pdf.

Then find the cdf.

Solution: $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} k|x|e^{-x^2}\,dx = \lim_{b\to-\infty}\int_{b}^{0} k|x|e^{-x^2}\,dx$$

$$+ \lim_{c\to\infty}\int_{0}^{c} k|x|e^{-x^2}\,dx.$$

$$\int_{b}^{0} k|x|e^{-x^2}\,dx$$

$$= \int_{b}^{0} k(-x)\,e^{-x^2}\,dx \quad \left[\; b<0 \text{ so } \cancel{x<b<0} \;\; b<x<0 \text{ and } |x| = -x \;\right]$$

$$= -k\int_{b}^{0} x e^{-x^2}\,dx$$

$-x^2 = u$.
$-2x\,dx = du$.
$-x\,dx = \frac{du}{2}$.
$x=0 \Rightarrow u=0$
$x=b \Rightarrow u=-b^2$

Note, we get.
$$\boxed{\int x e^{-x^2}\,dx = -\frac{e^{-x^2}}{2} + C}$$

Substituting

$$k\int_{-b^2}^{0} e^{u}\,\frac{du}{2}.$$

$$= \frac{k}{2} \; e^u \Big|_{-b^2}^{0} = \frac{k}{2}\left(e^0 - e^{-b^2}\right) = \frac{k}{2}\left(1 - e^{-b^2}\right)$$

Then $\displaystyle\lim_{b \to -\infty} \int_b^0 k|x| e^{-x^2} dx = \lim_{b \to -\infty} \frac{k}{2}\left(1 - e^{-b^2}\right)$

$$b \longrightarrow -\infty \Rightarrow b^2 \longrightarrow \infty$$
$$\Rightarrow -b^2 \longrightarrow -\infty$$
$$\Rightarrow e^{-b^2} \longrightarrow 0$$

$$\therefore \lim_{b \to -\infty} \int_b^0 k|x| e^{-x^2} = \frac{k}{2} \quad \text{--- ①}$$

Similarly

$$\lim_{c \to \infty} \int_0^c k|x| e^{-x^2} = \frac{k}{2} \quad \text{--- ②}.$$

$$\therefore \cancel{\int_{-\infty}^{\infty} \frac{k}{1+|x|^2} dx} \quad \int_{-\infty}^{\infty} k|x| e^{-x^2} dx = ① + ②$$
$$= \cancel{2k}\, k$$

$\shortparallel$

1 (pdf).

~~But~~

$\Rightarrow \cancel{2}k = 1 \Rightarrow \boxed{k = \cancel{1/2}\ 1}$

cdf: $F(x) = \cancel{\int_{-\infty}^{x} \frac{1}{2}|y| e^{-y^2} dx}$

$$= \int_{-\infty}^{x} \cancel{\tfrac{1}{2}}|y| e^{-y^2} dy.$$

$\boxed{\text{Breaks at } 0}$

If $x \le 0$. Then.

$$F(x) = \int_{-\infty}^{x} |y| e^{-y^2} dy = \int_{-\infty}^{x} (-y) e^{-y^2} dy \quad \left[\begin{array}{l} x \le 0 \\ \Rightarrow -\infty < y \le 0 \\ |y| = -y \end{array}\right]$$

$$= -\int_{-\infty}^{x} y e^{-y^2} dy$$

$$= \lim_{b \to -\infty} -\int_{b}^{x} y e^{-y^2} dy$$

$$= \lim_{b \to -\infty} \int_{-b^2}^{-x^2} e^{u} \frac{du}{2}$$

$$= \lim_{b \to -\infty} \frac{1}{2} \left(e^{-x^2} - e^{-b^2}\right) = \boxed{\frac{1}{2} e^{-x^2}}$$

Note $F(0) = \int_{-\infty}^{0} |y| e^{-y^2} dx = \frac{1}{2}$.

If $x \geq 0$ Then

$$F(x) = \int_{-\infty}^{x} |y| e^{-y^2} dy = \int_{-\infty}^{0} |y| e^{-y^2} dy = \frac{1}{2}.$$

$$+ \int_{0}^{x} |y| e^{-y^2} dy.$$

$$= \frac{1}{2} + \int_{0}^{x} |y| e^{-y^2} dy.$$

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$x \geq 0 \Rightarrow |y| = y \quad (0 \leq y \leq x)$

$$\int_0^x |y| e^{-y^2} dy = \int_0^x y e^{-y^2} dy$$

$$= -\int_0^{-x^2} e^u \frac{du}{2} \quad \left(\begin{matrix}\text{Same} \\ \text{substitution}\end{matrix}\right)$$

$$= -\frac{1}{2}\left(e^{-x^2} - e^0\right) = -\frac{1}{2}\left(e^{-x^2} - 1\right)$$

$$= \frac{1}{2}\left(1 - e^{-x^2}\right)$$

$\boxed{x \geq 0}$

$$\therefore F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_0^x f(x)\,dx$$

$$= \frac{1}{2} + \frac{1}{2}\left(1 - e^{-x^2}\right) = 1 - \frac{1}{2}e^{-x^2}$$