More techniques of Integration

13th February

Last class we learnt how to integrate $\sin^n(x)\cos^m(x)\,dx$

Now we will consider $\int \tan^m(x)\sec^n(x)\,dx$.

We will not discuss these but $\int \cot^m(x)\,\mathrm{cosec}^n(x)\,dx$ will be similar.

$\int \sec^3(x)\,dx$

Substituting $\longrightarrow$ | $u = \sec(x)$, $du = \sec(x)\tan(x)\,dx$ | $dv = \sec^2(x)\,dx$, $v = \int \sec^2(x)\,dx = \tan(x)$

$$\int u\,dv$$
$$= uv - \int v\,du \quad [\text{IBP}]$$
$$= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\,dx$$
$$= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,dx \quad \cdots \text{(1)}$$

Now

$$\int \sec(x)\tan^2(x)\,dx$$
$$= \int \sec(x)(\sec^2(x) - 1)\,dx$$
$$= \int \sec^3(x)\,dx - \int \sec(x)\,dx$$
$$= \int \sec^3(x)\,dx - \ln|\sec(x) + \tan(x)|$$

Substituting in (1), we get

$$\int \sec^3(x)\,dx = \sec(x)\tan(x) - \int \sec^3(x)\,dx + \ln|\sec(x)+\tan(x)| + C$$

$$\Rightarrow 2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x)+\tan(x)| + C$$

$$\Rightarrow \int \sec^3(x)\,dx = \frac{1}{2}\left\{\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\right\} + C.$$

Technique: 1) $\int \tan^{m}(x)\sec^{n}(x)\,dx$ $\boxed{n \text{ odd} ; m \text{ even}}$

- Rewrite $\tan^{m}(x)$ is terms of $\sec x$ (Use $\tan^2(x) = \sec^2(x) - 1$)

- Convert to a type $\int \sec^{r}(x)\,dx$ r-even.

- Now use IBP
  $u = \sec^{r-2}(x)$ $\quad dv = \sec^2(x)\,dx$.

- You will get an expression that contains. $\int \sec^{r-2}(x)\tan^2(x)\,dx$.
  Again use $\tan^2(x) = \sec^2(x) - 1$

- get $\int \sec^{r}(x)\,dx$ in terms of $\int \sec^{r-2}(x)\,dx$

- Repeat till you are down to $\sec(x) / \sec^2(x)$

Note,

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C$$

$$\int \tan(x)\,dx = \ln|\sec(x)| + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C.$$

→ Also for $\csc(x)$ instead of $\sec(x)$ and $\cot(x)$ instead of $\tan(x)$

1) ~~n $\int \tan^n$~~

2) <u>n even ($n=0$ as well)</u> $\int \tan^n(x) \sec^m(x)\,dx$

even

— Rewrite $\sec^{n-2}(x)$ in terms of $\tan(x)$
($\sec^2(x) = 1 + \tan^2(x)$)

— Then use substitution $u = \tan(x)$.

— $\int \tan^m(x) \sec^n(x) = \int \tan^m(x) \underbrace{\sec^{n-2}(x)}_{\text{Rewrite in } \tan(x)} \underbrace{\sec^2(x)\,dx}_{du}$.

<u>Easy case:</u> $\int \sec^2(x)\,dx$

3) <u>m odd</u>. $\int \tan^n(x) \sec^m(x)\,dx$.

even.

—Rewrite $\tan^{m-1}(x)$ in terms of $\sec(x)$.
($\tan^2(x) = \sec^2(x) - 1$

— Use $u = \sec(x)$
$du = \sec(x)\tan(x)\,dx$

~~$\int \tan$~~ <u>Example:</u> $\int \boxed{x^2} \tan^3(x^3) \sec(x^3) \boxed{dx}$.

| $\theta = x^3$
| $d\theta = 3x^2\,dx$
| $\frac{d\theta}{3} = \boxed{x^2\,dx}$

Substituting

~~$x^3 = \theta$~~ $= \int \tan^3(\theta) \sec(\theta) \frac{d\theta}{3}$

$= \frac{1}{3}\int \tan^2\theta \cdot \underline{\tan\theta \sec\theta\,d\theta}$

$= \frac{1}{3}\int (\sec^2\theta - 1) \tan\theta \sec\theta\,d\theta$

| $u = \sec\theta$
| $du = \sec\theta \tan\theta\,d\theta$

Substituting

~~$= \frac{1}{3}\int \tan\theta \sec^3\theta\,d\theta - \int \tan\theta\sec\theta\,d\theta$~~

$= \frac{1}{3}\int (u^2 - 1)\,du$.

$$= \frac{1}{3}\left[\frac{u^3}{3} - u\right] + C$$

$$= \frac{1}{3}\left[\frac{\sec^3(\theta)}{3} - \sec(\theta)\right] + C \quad [u = \sec\theta]$$

$$= \frac{1}{3}\left[\frac{\sec^3(x^3)}{3} - \sec(x^3)\right] + C \quad [\theta = x^3]$$

$\square$

$$\int \tan^n(x)\sec^m(x)\,dx = \int \tan^{n-1}(x)\sec^{m-1}(x)\,\underbrace{\tan(x)\sec(x)\,dx}_{du}$$

$\tan^{n-1}(x) \downarrow$ write in terms $\sec(x)$.

Exercise:

~~Sub~~ Integrate: $\displaystyle\int \frac{\sin^3(x)}{\sqrt{\cos(x)}}\,dx$. $\qquad \displaystyle\int \sin^n(x)\cos^m(x)$

$\boxed{n \text{ - is odd}}$.

$$= \int \frac{1}{\sqrt{\cos(x)}}\sin^2(x)\,\underbrace{\sin(x)\,dx}$$

$\sin^2(x) \downarrow 1 - \cos^2(x)$

$\sin(x)\,dx \downarrow$ substitute $u = \cos(x)$, $du = -\sin(x)\,dx$

Upshot: ~~The~~ These techniques work even if ~~n~~ one of n and m is a fraction.

Substitution (Trig)

$$\int (x^2+2x+3)^{1/2}\,dx \qquad \left(\begin{array}{l} x^2+2x+3 = u \\ \Rightarrow \boxed{(2x+2)\,dx = du} \end{array}\right)$$

Complete the square. next??

$$= \int \left\{ (x^2+2\cdot 1x+1^2)+3 - 1^2 \right\}^{1/2} dx$$

$$= \int \left\{ \boxed{(x+1)^2} + \boxed{2} \right\}^{1/2} dx$$ ~~What if~~

What if it was $\int (t^2+1)^{1/2}\,dt$?

Substitute $(x+1) = \sqrt{2}\tan\theta \iff \theta = \arctan\left(\frac{x+1}{\sqrt{2}}\right)$

$$dx = \frac{d}{dx}(x+1)\,dx = \frac{d}{d\theta}(\sqrt{2}\tan\theta)\,d\theta = \sqrt{2}\sec^2\theta\,d\theta$$

We get

$$\int \left\{ (\sqrt{2}\tan\theta)^2 + 2 \right\}^{\frac{1}{2}} \sqrt{2}\sec^2\theta\,d\theta$$

$$= \int \left\{ 2\tan^2\theta + 2 \right\}^{1/2} \sqrt{2}\sec^2\theta\,d\theta$$

~~$= \int \sqrt{2}\,\{\tan^2 +$~~

$$= \int \left\{ 2(\tan^2\theta + 1) \right\}^{1/2} \sqrt{2}\sec^2\theta\,d\theta$$

$$= \int \sqrt{2}\,(\sec^2\theta)^{1/2}\,\sqrt{2}\sec^2\theta\,d\theta$$

$$= 2\int \sec^3\theta\,d\theta$$

From a previous problem, we get

$$2\int \sec^3\theta \, d\theta = \{\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|\} + C$$

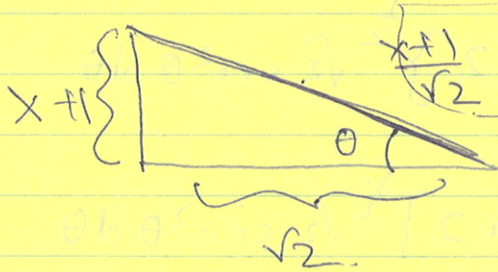
~~Now put back~~ $\cancel{\theta = \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)}$, ~~to get~~

$$\cancel{\left\{ \sec\left(\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right) \tan\left(\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right) + \ln\left|\sec\left(\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right.\right.}$$

~~Now put back~~ $\cancel{\theta = \arctan\left(\frac{x+1}{\sqrt{2}}\right)}$ ~~to get~~

$$\cancel{\left\{ \sec\left(\arctan\left(\frac{x+1}{\sqrt{2}}\right)\right) \tan\left(\arctan\left(\frac{x+1}{\sqrt{2}}\right)\right) + \ln\left|\sec\left(\arctan\left(\frac{x+1}{\sqrt{2}}\right)\right) + \tan\left(\arctan\left(\frac{x+1}{\sqrt{2}}\right)\right)\right|\right.}$$

If $x > 0$.



$$\boxed{\frac{x+1}{\sqrt{2}} = \tan\theta} = \frac{\text{height}}{\text{base}}$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{base}}$$

$\cancel{\tan\theta = \frac{x+1}{\sqrt{2}} \quad \sec\theta = \sqrt{}}$

$$= \frac{\sqrt{(x+1)^2 + (\sqrt{2})^2}}{\sqrt{2}}$$

$$= \frac{\sqrt{x^2+2x+3}}{\sqrt{2}}$$

Put back $\sec\theta = \frac{\sqrt{x^2+2x+3}}{\sqrt{2}}$ and $\tan\theta = \frac{x+1}{\sqrt{2}}$

to get the answer.

Substitutions

Form	Substitution	Only if
1) $\sqrt{(a^2 - x^2)}$	$x = a\sin\theta$	$-a \le x \le a$ $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
2) $a^2 + x^2$ → previous problem	$x = a\tan\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
3) $\sqrt{(x^2 - a^2)}$	$x = a\sec\theta$	$x \ge a$ $0 \le \theta < \frac{\pi}{2}$ or $x \le -a$ $\frac{\pi}{2} < \theta < \pi$

Step.

- Complete the square
- Identify the form.
- Make the substitution from table.
- Integrate using trig. integrals. $\int \sin^m(x)\cos^n(x)dx$, $\int \tan^n(x)\sec^m(x)dx$

a^2 could be 9, 15, π

x could be $x+1$, $x+10$, ~~$x-15$~~

x^2 ~~could be~~

$$\int \frac{x\,dx}{\sqrt{x^2+1}} \longrightarrow \text{No need for } \boxed{x = \tan\theta}$$

Try $u = x^2+1$.

Note: also $-(a^2-x^2) = x^2 - a^2$. So.

So it is important to see what value x can take., e.g.

$\sqrt{x^2-a^2}$ indicates $x \ge a$ & $x \le -a$, so $x = a\sec\theta$

and $\sqrt{a^2-x^2}$ indicates
$-a \le x \le a$ so $x = a\sin\theta$

(8).

$$\int_{3/2}^{\frac{3\sqrt{3}}{2}} \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

$9 = 3^2$. the form is.

$$\sqrt{9-x^2} = \sqrt{3^2-x^2} \qquad \text{so } -3 \le x \le 3$$

Let $x = 3\sin\theta$

$$dx = \cancel{3\cos\theta\, d\theta}\ \frac{d(3\sin\theta)}{d\theta}\,d\theta = 3\cos\theta\,d\theta$$

~~$x = \frac{3}{2} \Rightarrow 3\sin\theta = \frac{3}{2} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta =$~~

~~Substituting we get,~~

$$x = \frac{3}{2} \Rightarrow 3\sin\theta = \frac{3}{2} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$x = \frac{3\sqrt{3}}{2} \Rightarrow 3\sin\theta = \frac{3\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

Substituting, we get,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\sin^2\theta}{\sqrt{9(1-\sin^2\theta)}}\,3\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\cdot\sin^2\theta}{\sqrt{9\cos^2\theta}}\,3\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\sin^2\theta}{\cancel{9}\cancel{3\cos\theta}} \cancel{3\cos\theta}\, d\theta \qquad = 9\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\, d\theta$$

$$= \frac{9}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin^2\theta\, d\theta$$

$$= \frac{9}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\cos 2\theta)\, d\theta$$

$$\cancel{= \frac{9}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta}$$

$$= \frac{9}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta - \frac{9}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 2\theta\, d\theta$$

$$= \frac{9}{2}\, \theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{9}{2}\, \frac{\sin 2\theta}{2} \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{9}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \frac{9}{2}\left(\frac{\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)}{2}\right)$$

$$= \frac{\cancel{9}^{3}}{2}\left(\frac{\pi}{\cancel{6}_{2}}\right) - \frac{9}{4}\left(\frac{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{2}\right)$$

$$= \frac{3\pi}{4}.$$