

Expectation and Variance

① 13th March, 2014

Recall Random Variable $X(\text{situation}) = \text{value (heads or tails)}$
(tossed coin)

We are interested in

$$P_X(a \leq X \leq b)$$

probability of X lying between a and b .

$$P_X(a \leq X \leq b) = \int_a^b f(t) dt$$

p.d.f. Properties.

$$= F(b) - F(a)$$

$$1) \int_{-\infty}^{\infty} f(t) dt = 1$$

$$2) f(t) \geq 0$$

F - c.d.f. \leftrightarrow

Properties

1) $\lim_{x \rightarrow \infty} F(x) = 1$

Definition: $F(x) = \int_{-\infty}^x P_X(X \leq t)$

Properties:

$$1) 0 \leq F(x) \leq 1$$

2) F is non-decreasing

$$\lim_{x \rightarrow \infty} F(x) = 1$$

$$\lim_{x \rightarrow -\infty} F(x) = 0$$

Relation between f and F .

$$F(x) = \int_{-\infty}^x f(t) dt$$

$$f(x) = F'(x)$$

(Mean) \Rightarrow (Average) - (Expected Value) of X .

$E(X) = \int_{-\infty}^{\infty} x f(x) dx$ - What value do we expect to happen?
(Law of Large numbers)
Can give us a

$x \rightarrow$ Value.

$f(x) \rightarrow$ weight

$\int_{-\infty}^{\infty} x f(x) dx \rightarrow$ weighted average.

wrong notion?

~~If heads for~~

~~If $P(X=1)$~~

~~$P(X=1) = \frac{1}{2}$~~

~~$P(X=-1) = \frac{1}{2}$~~

~~Then $E(X) = 0$~~

~~But $P(X=0) = 0$~~

Suppose

$f(x) = \begin{cases} \frac{1}{2} & -2 < x < -1 \\ \frac{1}{2} & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Then $E(X) = 0$ But $f(0) = 0$

Variance of X

$Var(X) = \int_{-\infty}^{\infty} (x - E(x))^2 f(x) dx$ -> Always positive
non-negative

Standard Deviation of X

$\sigma(x) = \sqrt{Var(X)}$

To find mean
 \rightarrow integrate

To find variance,
 \rightarrow find mean
 \rightarrow integrate

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Qn. Let $f(x) = \begin{cases} \frac{\sin x}{2} & 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$

Find Check that this is a probability density function and to find the mean and variance.

Solution: We need to check.

1) $f(x) \geq 0$

2) $\int_{-\infty}^{\infty} f(x) dx = 1$

to prove that f is a pdf.

Checking 1)

If $x \leq 0$ or $x \geq \pi$ Then $f(x) = 0$.

If $0 < x < \pi$ then $f(x) = \frac{\sin(x)}{2} > 0$

$\therefore f(x) \geq 0$ for all x .

Checking 2) $\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{\sin(x)}{2} dx$

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Now $f(x) = 0$ when $x \leq 0$ or $x \geq \pi$.

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \int_0^{\pi} f(x) dx = \int_0^{\pi} \frac{\sin(x)}{2} dx.$$

Another way to see this.

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\pi} f(x) dx + \int_{\pi}^{\infty} f(x) dx$$

$$\begin{aligned} \int_0^{\pi} \frac{\sin x}{2} dx &= \frac{1}{2} \int_0^{\pi} \sin x dx = \frac{1}{2} (-\cos x) \Big|_0^{\pi} \\ &= \frac{1}{2} \{(-\cos(\pi)) - (-\cos(0))\} \\ &= \frac{1}{2} \{(-(-1)) - (-1)\} \\ &= \frac{1}{2} (1+1) = 1. \end{aligned}$$

$$FE(x) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^0 x f(x) dx + \int_0^{\pi} x f(x) dx + \int_{\pi}^{\infty} x f(x) dx.$$

$$= \frac{1}{2} \int_0^{\pi} x \sin x \, dx$$

$$u = x \quad dv = \sin x \, dx$$

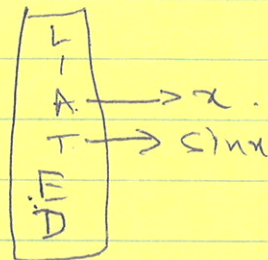
$$du = dx \quad v = \int \sin x \, dx = -\cos x$$

Substituting, we get, (forget about limits)

$$\frac{1}{2} \int u \, dv$$

IBP

$$= \frac{1}{2} \{ uv - \int v \, du \}$$



$$= \frac{1}{2} x (-\cos x) - \frac{1}{2} \int (-\cos x) \, dx$$

$$= -\frac{1}{2} x \cos x + \frac{1}{2} \int \cos x \, dx$$

$$= -\frac{1}{2} x \cos x + \frac{1}{2} \sin(x) + c$$

$$\therefore \frac{1}{2} \int_0^{\pi} x \sin x \, dx$$

$$= \left(-\frac{1}{2} x \cos x + \frac{1}{2} \sin x \right) \Big|_0^{\pi}$$

$$= -\frac{1}{2} \pi \cos(\pi) + \frac{1}{2} \sin(\pi)$$

$$- \frac{1}{2} (-0 \cos(0) + \sin(0))$$

$$= -\frac{1}{2} \pi (-1) + \frac{1}{2} 0$$

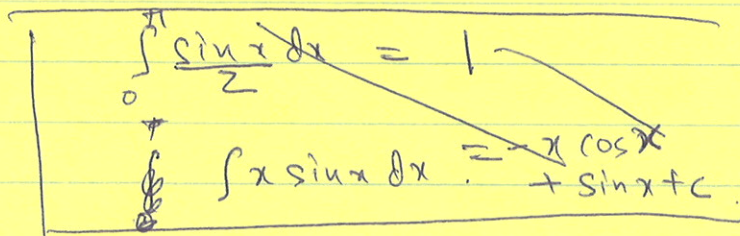
$$- \frac{1}{2} (-0 + 0)$$

$$= \frac{\pi}{2}$$

$$1) \int_0^{\pi} \sin x \, dx = 2$$

$$2) \int x \sin x \, dx = -x \cos x + \sin x + c$$

$$3) \int_0^{\pi} x \sin x \, dx = \pi$$



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Variance

$$\int_{-\infty}^{\infty} x^2 f(x) dx = \int_{-\infty}^0 x^2 f(x) dx + \int_0^{\infty} x^2 f(x) dx$$

$$= \int_0^{\pi} \frac{x^2 \sin x}{2} dx$$

$$= \frac{1}{2} \int_0^{\pi} x^2 \sin x dx$$

$$\int x^2 \sin x dx$$

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Variance:

$$\int_{-\infty}^{\infty} (x - E(x))^2 f(x) dx$$

$$= \int_{-\infty}^0 (x - E(x))^2 f(x) dx$$

$$+ \int_0^{\pi} (x - E(x))^2 f(x) dx$$

$$+ \int_{\pi}^{\infty} (x - E(x))^2 f(x) dx$$

$$= \int_0^{\pi} (x - \frac{\pi}{2})^2 \frac{\sin x}{2} dx$$

$$= \frac{1}{2} \int_0^{\pi} (x^2 + \frac{\pi^2}{4} - \pi x) \sin x dx$$

$$= \frac{1}{2} \left[\int_0^{\pi} x^2 \sin x dx + \int_0^{\pi} \frac{\pi^2}{4} \sin x dx - \int_0^{\pi} \pi x \sin x dx \right]$$

$$= \frac{1}{2} \left[\int_0^{\pi} x^2 \sin x dx + \frac{\pi^2}{4} (2) - \pi (\pi) \right] \quad \left[\begin{array}{l} \text{Using (1)} \\ \text{and} \\ \text{(2)} \end{array} \right]$$

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matrix

$$A(x) = \begin{pmatrix} x & 1 \\ 1 & x \end{pmatrix}$$

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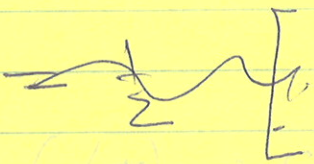
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1
2
3

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$$\int x^2 \sin x \, dx =$$

Substituting \longrightarrow

$$\int u \, dv \quad (u) \frac{d}{dx} (v) = (u)' v + (u) v'$$

$u = x^2$	$dv = \sin x \, dx$	$\xrightarrow{A} x^2$
$du = 2x \, dx$	$v = \int \sin x \, dx$	$\xrightarrow{B} \sin x$
	$= -\cos x$	

IBP

$$uv - \int v \, du$$

$$= x^2 (-\cos x) - \int (-\cos x) 2x \, dx$$

$$= -x^2 \cos x + 2 \int (\cos x) x \, dx$$

Substituting \longrightarrow

$u = x$	$dv = \cos x \, dx$
$du = dx$	$v = \int \cos x \, dx$
	$= \sin x$

$$= -x^2 \cos(x) + 2 \int u \, dv$$

IBP

$$-x^2 \cos x + 2 \{ uv - \int v \, du \}$$

$$= -x^2 \cos x + 2 \left\{ x \sin x - \int \sin(x) \, dx \right\}$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos(x) + C$$

$$\int_0^\pi x^2 \sin x \, dx = \left[-x^2 \cos x + 2x \sin x + 2 \cos(x) \right]_0^\pi$$

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$$= (-\pi^2 \cos(\pi) + 2\pi \sin(\pi) + 2 \cos(\pi))$$

$$= (-\pi^2 \cos(\pi) + 2\pi \sin(\pi) + 2 \cos(\pi))$$

$$= (-0^2 \cos(0) + 2 \cdot 0 \cdot \sin(0) + 2 \cos(0))$$

$$= (+\pi^2 + 2\pi(0) + 2(-1))$$

$$= (-0 + 0 + 2(1))$$

$$= \pi^2 - 2 - 2 = \pi^2 - 4$$

\therefore Variance =

$$\frac{1}{2} \left[\pi^2 - 4 + \frac{\pi^2}{2} - \pi^2 \right]$$

$$= \frac{1}{2} \left(\frac{\pi^2}{2} - 4 \right) = \frac{\pi^2}{4} - 2 \quad \text{Ans}$$