Expectation and Variance

13th March, 2014

Recall Random Variable $X(\quad) = $ value. (heads or tails)

$\uparrow$ situation (toss of coin)

∴ We are interested in

$$\mathbb{P}r(a \le X \le b)$$

probability of X lying between a and b.

$$\mathbb{P}r(a \le X \le b) = \int_a^b f(t)\,dt$$

f — p.d.f.

Properties.

1) $\int_{-\infty}^{\infty} f(t)\,dt = 1$

2) $f(t) \ge 0$

~~3)~~

$$= F(b) - F(a)$$

F — c.d.f. ~~F(x)~~

~~Properties~~

~~1) Lim~~ ~~Def~~
~~x→∞~~

Definition: $F(x) = \mathbb{P}r(X \le x)$

Properties:

1) $0 \le F(x) \le 1$

2) F is non-decreasing

3) $\lim_{x \to \infty} F(x) = 1$

$\lim_{x \to -\infty} F(x) = 0$

Relation between f and F.

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$f(x) = F'(x)$$

(Mean) → (Average) - (Expected Value) of X.

$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$ — What value do we expect to happen?

↓ (Law of Large numbers)

Can give us a wrong notion?

$x \to$ Value

$f(x) \to$ weight

$\int_{-\infty}^{\infty} x f(x)\, dx \to$ weighted average.

~~If heads for~~

If ~~$\mathbb{P}(X=)$~~

$\mathbb{P}(X=1) = \frac{1}{2}$

$\mathbb{P}(X=-1) = \frac{1}{2}$

Then $\mathbb{E}(X) = 0$

But $\mathbb{P}(X=0) = 0$

Suppose

$$f(x) = \begin{cases} \frac{1}{2} & -2 < x < -1 \\ \frac{1}{2} & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Then $\mathbb{E}(X) = 0$ But $f(0) = 0$

Variance of X

$$Var(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(x))^2 f(x)\, dx \to$$ Always ~~positive~~ non-negative.

Standard Deviation of X

$$\sigma(x) = \sqrt{Var(X)}$$

To find mean
→ integrate

To find variance.
→ find mean
→ integrate.

Qn. Let $f(x) = \begin{cases} \frac{\sin x}{2} & 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$

~~Find~~ Check that this is a probability density function and find the mean and ~~variab~~ variance.

Solution: We need to check

1> $f(x) \geq 0$

2> $\int_{-\infty}^{\infty} f(x)\, dx = 1$

to prove that f is a pdf.

Checking 1>

If $x \leq 0$ or $x \geq \pi$ Then $f(x) = 0$.

If $0 < x < \pi$ then $f(x) = \sin(x) \geq 0$

$\therefore$ $f(x) \geq 0$ for all x.

Checking 2> $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \cancel{\sin(x)\, dx}$

Now $f(x) = 0$ when $x \le 0$ or $x \geq \pi$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\pi} f(x)\,dx = \int_{0}^{\pi} \frac{\sin(x)}{2}\,dx.$$

[Another way to see this.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} \cancel{f(x)}^{\,0}\,dx + \boxed{\int_{0}^{\pi} f(x)\,dx} + \int_{\pi}^{\infty} \cancel{f(x)}^{\,0}\,dx.$$
]

$$\int_{0}^{\pi} \frac{\sin x}{2}\,dx. = \frac{1}{2}\int_{0}^{\pi} \sin x\,dx = \frac{1}{2}(-\cos x)\Big|_{0}^{\pi}$$

$$= \frac{1}{2}\Big\{\big(-\cos(\pi)\big) - \big(-\cos(0)\big)\Big\}$$

$$= \frac{1}{2}\Big\{\big(-(-1)\big) - (-1)\Big\}$$

$$= \frac{1}{2}(1+1) = 1.$$

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} \cancel{x f(x)}^{\,0}\,dx + \int_{0}^{\pi} x f(x)\,dx. + \int_{\pi}^{\infty} \cancel{x f(x)}^{\,0}\,dx.$$

$$= \frac{1}{2}\int_0^{\pi} x \sin x \, dx.$$

$\cancel{\int x \sin x \, dx}$.

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = \int \sin x \, dx = -\cos x.$

Substituting, we get, (forget about limits)

$$\frac{1}{2}\int u \, dv$$

IBP

$$= \frac{1}{2}\left\{uv - \int v \, du\right\}$$

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A $\rightarrow x$.
T $\rightarrow \sin x$
E
D

$$= \cancel{\sin x}\ \frac{1}{2}x(-\cos x) - \frac{1}{2}\int(-\cos x)\,dx.$$

$$= -\frac{1}{2}x\cos x + \frac{1}{2}\int \cos x \, dx$$

$$= -\frac{1}{2}x\cos x + \frac{1}{2}\sin(x) + c$$

$$\therefore \frac{1}{2}\int_0^{\pi} x \sin x \, dx$$

$$= \left(-\frac{1}{2}x\cos x + \frac{1}{2}\sin x\right)\Big|_0^{\pi}$$

$$= \cancel{x\cos}\ -\frac{1}{2}\pi\cos(\pi) + \frac{1}{2}\sin(\pi) - \frac{1}{2}\left(-0\ \cancel{\sin}\cos(0) + \sin(0)\right)$$

$$= -\frac{1}{2}\pi(-1) + \frac{1}{2}0 - \frac{1}{2}(-0 + 0)$$

$$= \frac{\pi}{2}$$

$$\boxed{\int_0^{\pi} \frac{\sin x}{2}\,dx = 1 \qquad \int x \sin x \, dx = -x\cos x + \sin x + c.}$$

1) $\int_0^{\pi} \sin x \, dx = 2$

2) $\int x \sin x \, dx = -x\cos x + \sin x + c$.

3) $\int_0^{\pi} x \sin x \, dx = \pi$.

Variance

$$\int_{-\infty}^{\infty} x^2 \, f(x)\, dx =$$

$$= \int_{-\infty}^{0} x^2 \underbrace{\cancel{f(x)}}_{0}\, dx + \int_{0}^{\pi} x^2 \, f(x)\, dx$$

$$+ \int_{\pi}^{\infty} x^2 \underbrace{\cancel{f(x)}}_{0}\, dx$$

$$= \int_{0}^{\pi} \frac{x^2 \sin x}{2}\, dx.$$

$$= \frac{1}{2} \int_{0}^{\pi} x^2 \sin x \, dx.$$

$$\int x^2 \sin x \, dx.$$

Variance:

$$\int_{-\infty}^{\infty} \left(x - \mathbb{E}(X)\right)^2 f(x)\, dx$$

$$= \int_{-\infty}^{0} \cancelto{0}{\left(x - \mathbb{E}(X)\right)^2 f(x)}\, dx$$

$$+ \int_{0}^{\pi} \left(x - \mathbb{E}(X)\right)^2 f(x)\, dx$$

$$+ \int_{\pi}^{\infty} \cancelto{0}{\left(x - \mathbb{E}(X)\right)^2 f(x)}\, dx$$

$$= \int_{0}^{\pi} \left(x - \frac{\pi}{2}\right)^2 \frac{\sin x}{2}\, dx$$

$$= \frac{1}{2}\int_{0}^{\pi} \left(x^2 + \frac{\pi^2}{4} - \pi x\right) \sin x \cdot dx$$

$$= \frac{1}{2}\left[\int_{0}^{\pi} x^2 \sin x\, dx + \int_{0}^{\pi} \frac{\pi^2}{4} \sin x\, dx - \int_{0}^{\pi} \pi x \sin x\, dx\right]$$

$$= \frac{1}{2}\left[\int_{0}^{\pi} x^2 \sin x\, dx + \frac{\pi^2}{4}(2) - \pi(\pi)\right] \quad \text{(4)}$$

[Using (1) and (2)]

$$\int x^2 \sin x \, dx =$$

Substituting $\longrightarrow$ $u = x^2$, $dv = \sin x \, dx$; $du = 2x\,dx$, $v = \int \sin x \, dx = -\cos x$

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A $\to x^2$
T $\to \sin x$
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P

$$\int u \, dv$$

$\underline{\underline{\text{IBP}}}$ $uv - \int v \, du$.

$$= x^2(-\cos x) - \int (-\cos x)\, 2x \, dx.$$

$$= -x^2 \cos x + 2\int (\cos x)\, x \, dx.$$

Substituting $\longrightarrow$ $u = x$, $dv = \cos x \, dx$; $du = dx$, $v = \int \cos x \, dx = \sin x$.

$$= -x^2 \cos(x) + 2\int u \, dv.$$

$\underline{\underline{\text{IBP}}}$ $-x^2 \cos x + 2\{uv - \int v \, du\}$

$$= -x^2 \cos x + 2\left\{x \sin x - \int \sin(x)\, dx\right\}.$$

$$= -x^2 \cos x + 2x \sin x + 2\cos(x) + C$$

$$\int_0^{\pi} x^2 \sin x \, dx = \left(-x^2 \cos x + 2x \sin x + 2\cos x\right)\Big|_0^{\pi}.$$

$$\cancel{= (-\pi^2 \cos(\pi}$$

$$= \left(-\pi^2 \cos(\pi) + 2\pi \sin(\pi) + 2\cos(\pi)\right)$$

$$- \left(-0^2 \cos(0) + 2 \cdot 0 \cdot \sin(0) + 2\cos(0)\right)$$

$$= \left(+\pi^2 + 2\pi(0) + 2(-1)\right)$$

$$- \left(-0 + 0 + 2(1)\right)$$

$$= \pi^2 - 2 - 2 = \underline{\pi^2 - 4}.$$

$\therefore$ Variance =

$$\frac{1}{2}\left[\pi^2 - 4 + \frac{\pi^2}{2} - \pi^2\right]$$

$$= \frac{1}{2}\left(\frac{\pi^2}{2} - 4\right) = \frac{\pi^2}{4} - 2.$$ Ans