Partial derivatives ① 14th January

Notation:

- $\langle 2,3,4 \rangle$ is the same as $2\vec{i}+3\vec{j}+4\vec{k}$.
- contour line is the same as level lines.
- normal to the plane is perpendicular to plane
- Distance between $(5,6,7)$ and yz-plane.
  yz-plane is $x=0$
  $(5,6,7)$ lies on $x=5$.
  Distance between them is 5.

Partial Derivatives

$f(x)=x^2$
$f'(x)=2x$

$f(x)=ax^2$
$f'(x)=2ax^2$, if "a" is a constant.

Partial derivatives are derivatives of $f(x,y)$ when one of the variables is fixed and the other is assumed constant

Derivatives	Partial Derivatives
$f(x)$	$f(x,y)$.
$f'(a)$, $\frac{d}{dx}f(x)\big\|_{x=a}$	Notation: $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$
Derivative of f at a is $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ if it exists.	Definition: Partial derivatives of f at (a,b) ~~is~~ with respect to x is $f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$ ~~f_y~~ and with respect to y is $f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$ when it exists.
$f(x) = x^2$, $f'(a) = 2a$	$f(x) = x^2y+2$, $f_x = 2xy$, $f_y = x^2$.

The name of variables is not important. (X,Y) could be (P,Q) or (U,V) or anything. (What is a name?)

Alternative notation: $\frac{\partial f}{\partial x} = f_x$ $\quad \frac{\partial f}{\partial y} = f_y$.

Meaning: Suppose production P is a function of capital K and labour L. What is meant by $\frac{\partial P}{\partial K}$?

Solution: $\frac{\partial P}{\partial K}$ is the rate of increase of production with respect to capital keeping labour fixed = marginal production with respect to capital.

- $f(x,y) = x^2y + y^3 - x$. Find $f_x(1,1)$

Solution: $f_x = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial x}(y^3) - \frac{\partial}{\partial x}(x)$

constant

Y is fixed, so constant

$f_x = 2xy + 0 - 1 = 2xy - 1$

$f_x(1,1) = 2 - 1 = 1$.

- $f(x,y) = \sin(xy)$ Find $f_y(0,0)$

Solution: $f_y = \frac{\partial}{\partial y}(\sin(xy))$ → constant

X is fixed.

$f_y = \frac{\partial}{\partial y}(\sin(xy))$

$= \cos(xy)\frac{\partial}{\partial y}(xy)$. By Chain rule

X is a fixed number.

$= \cos(xy)\, x$

$= x\cos(xy)$.

$$f_y(0,0) = 0 \cdot \cos(0 \cdot 0) = 0 \cdot 1 = 0.$$

Second derivatives (derivatives of derivatives)

$$(f_x)_y = f_{xy} = \cancel{\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$(f_y)_x = f_{\underset{\text{first}}{y}\,\underset{\text{second}}{x}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

- Find all second derivatives of $f(x,y) = xy + e^{x+y}$ at $(1,0)$.

Key- first find derivative then plug $(1,0)$

Solution:
$$f_x = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(e^{x+y})$$
$$= y + e^{x+y}\left(\frac{\partial}{\partial x}(x+y)\right) \quad [\text{Chain Rule}]$$
$$= y + e^{x+y}(1) = y + e^{x+y}.$$

$$f_y = \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}(e^{x+y})$$
$$= x + e^{x+y}\left(\frac{\partial}{\partial y}(x+y)\right) = x + e^{x+y}(1) = x + e^{x+y}.$$
(Chain Rule).

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(y + e^{x+y}\right)$$

$$= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial x}\left(e^{x+y}\right)$$

$$= 0 + e^{x+y}\frac{\partial}{\partial x}(x+y) \quad \text{((Chain Rule)}$$

$$= e^{x+y}(1) = e^{x+y}.$$

$$\cancel{f_{xx}}\ (f_{xx}(1,0) = e^{1+0} = e$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(y + e^{x+y}\right)$$

$$= \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}\left(e^{x+y}\right)$$

$$= 1 + e^{x+y}\frac{\partial}{\partial y}(x+y)$$

$$= 1 + e^{x+y}(1) = 1 + e^{x+y}.$$

$$f_{xy}(1,0) = 1 + e^{1+0} = 1+e.$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(x + e^{x+y}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}\left(e^{x+y}\right)$$

$$= 1 + e^{x+y}\frac{\partial}{\partial x}(x+y)$$

$$= 1 + e^{x+y}(1) = 1 + e^{x+y}$$

$$f_{yx}(1,0) = 1 + e^{1+0} = 1+e.$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(x + e^{x+y}\right)$$

$$= \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}\left(e^{x+y}\right).$$

$$= 0 + e^{x+y}\frac{\partial}{\partial y}(x+y)$$

$$= \cancel{0 +} \; e^{x+y}(1)$$

$$= e^{x+y}.$$

$f_{yy}(1,0) = e^{1+0} = e.$

Note $f_{xy} = f_{yx}$. This is true in general.

~~Clairaut's~~ <u>Clairaut's Theorem</u>: If f_{xy} and f_{yx} are continuous then $f_{xy} = f_{yx}$

<u>Application</u>: Does there exist a function f such that $f_x = 2y$ and $f_y = 3x$.

<u>Solution</u>: Suppose there does exist ~~such~~ a function f such that

$$f_x = 2y \quad \text{and} \quad f_y = 3x.$$

Then $f_{xy} = 2$ and $f_{yx} = 3$

implying $f_{xy} \neq f_{yx}$. This contradicts Clairaut's Theorem. So there does not exist such a function !!