Local Maximas and Local Minimas

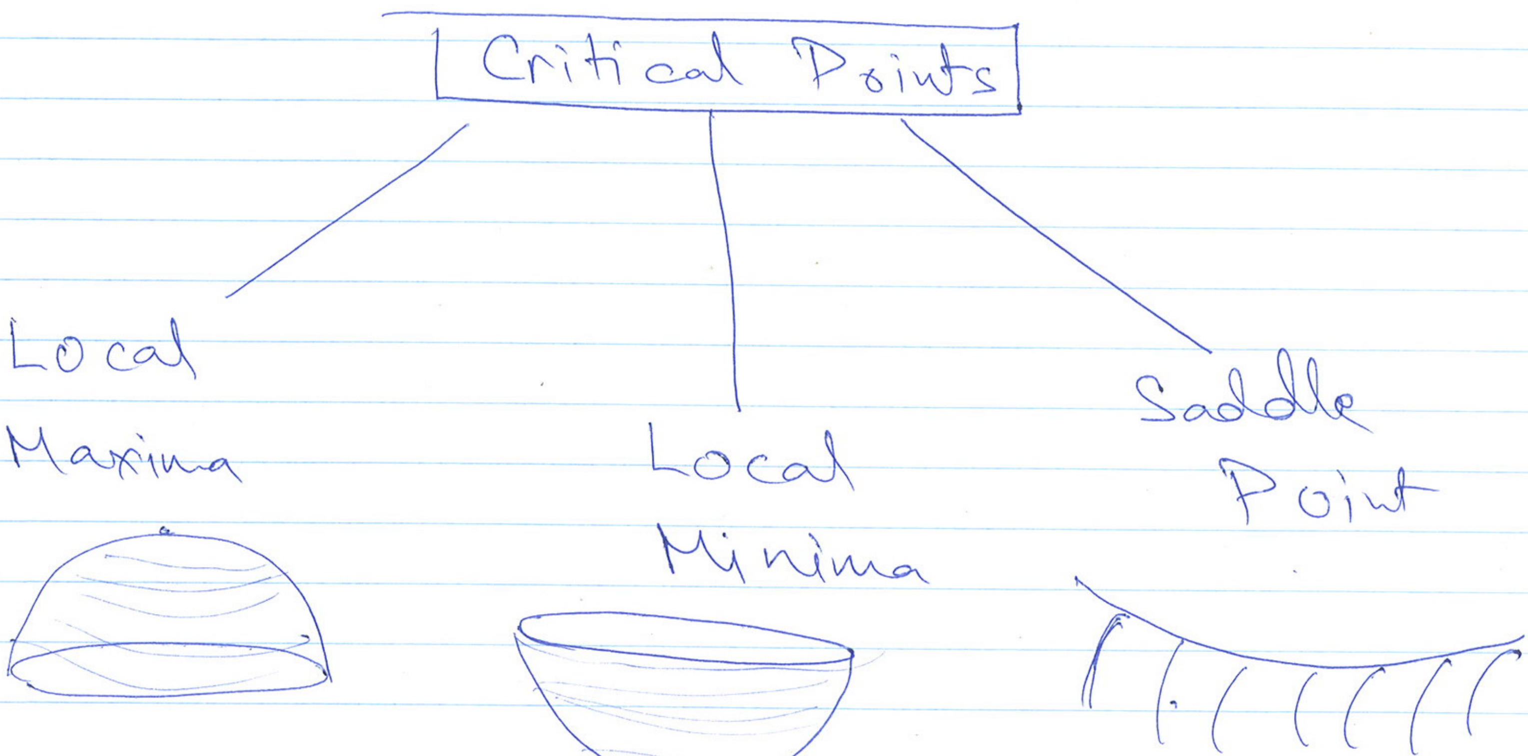
16th January

Critical Point: A point (a,b) is a critical point of f if either

- $f_x(a,b) = f_y(a,b) = 0$

or

- f_x or f_y do not exist at (a,b)



- f has a local <u>maxima</u> at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) near (a,b) in the domain.

- f has a local <u>minima</u> at (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) near (a,b) in the domain.

- f has a <u>saddle point</u>, ~~for (x,y)~~ at (a,b) if for points (x,y) near (a,b) there are some where $f(a,b) < f(x,y)$ and others where $f(a,b) > f(x,y)$

<u>Theorem:</u> If f has a local maximum or minimum at (a,b) and f_x, f_y exist the $f_x(a,b)=0$ and $f_y(a,b)=0$.

This is how maxima, minimas and critical points are related.

<u>Example:</u> find all critical points of f where $f(x,y) = ye^x - e^y$

<u>Solution</u>: • Find partial derivatives.

constant

$$f_x = \frac{\partial}{\partial x}(ye^x) - \frac{\partial}{\partial x}(e^y)$$

$$= ye^x$$

constant

$$f_y = \frac{\partial}{\partial y}(ye^x) - \frac{\partial}{\partial y}(e^y)$$

$$= e^x - e^y.$$

• Make $f_x, f_y = 0$ and solve for x and y.

$$f_x = 0 \Rightarrow ye^x = 0 \quad \text{--- (1)}$$

$$f_y = 0 \Rightarrow e^x - e^y = 0 \quad \text{--- (2)}.$$

Now e^x is never equal to 0.

So (1) implies $y = 0$.

Plugging $y = 0$ in (2), we get

$$e^x - e^0 = 0$$

$$\Rightarrow e^x - 1 = 0$$

$$\Rightarrow e^x = 1 = e^0$$

~~Taking loge ln on both sides:~~

$$\cancel{\Rightarrow \ln e^x = \ln e^0.}$$

~~By definition of ln~~. $x = 0$.

- ~~Make~~ The ~~points~~ found ~~satisfy $x=0, y=0$.~~
  ~~that~~ is $(0,0)$.

- Make sure the point is in the <u>domain</u> of f.

$(0,0)$ is ~~the~~ in the domain of f.

Thus $(0,0)$ is a critical point.

How to distinguish between maximas and minimas?

In 1 dimension
- $f'(x)=0$ $f''(x)<0$ Local Maxima ~~Minima~~
- $f''(x)=0$ $f''(x)>0$ Local Minima

~~$f''(x)=0$~~

In 2 dimensions?

Suppose $f_x(a,b)=f_y(a,b)=0$.

Let $D(a,b)=f_{xx}(a,b)\cdot f_{yy}(a,b)-[f_{xy}(a,b)]^2$.

This is called the <u>discriminant</u>

Second Derivative Test

1> If $D(a,b) > 0$ $f_{xx}(a,b) < 0$ then f has a local maximum at (a,b)

2> If $D(a,b) > 0$ $f_{xx}(a,b) > 0$ then f has a local minimum at (a,b)

3> If $D(a,b) < 0$ then f has a saddle point at (a,b).

4> If $D(a,b) = 0$, then inconclusive

Need $f_x(a,b) = f_y(a,b) = 0$ first

In the previous example

$f_x = ye^x$,

$f_y = e^x - e^y$ and the critical point was $(0,0)$.

$f_{xy} = \frac{\partial}{\partial y}(ye^x) = e^x \qquad f_{xy}(0,0) = 1$

$f_{xx} = \frac{\partial}{\partial x}(ye^x) = ye^x \qquad f_{xx}(0,0) = 0$

$f_{yy} = \frac{\partial}{\partial y}(e^x - e^y) = -e^y \qquad f_{yy}(0,0) = -1$

$D(0,0) = (f_{xx}^{(0,0)})(f_{yy}^{(0,0)}) - (f_{xy}^{(0,0)})^2 = (0)(-1) - 1^2 < 0$.

Thus $(0,0)$ is the saddle point for f.

<u>Technique</u>: 1) Find f_x, f_y.

2) Find points (a,b) where

$$f_x(a,b) = f_y(a,b) = 0.$$

3) Compute f_{xx}, f_{yy}, f_{xy} at ~~(a,b)~~ these points.

4) Compute D at these points.

5) Determine using chart.

~~$D(a,b) > 0$~~

~~$D(a,b) < 0$~~.

(a,b) critical point

- $D(a,b) > 0$
  - $f_{xx} > 0$: Local Minima
  - $f_{xx} < 0$: Local Maxima.
- $D(a,b) < 0$: Saddle point
- $D(a,b) = 0$: ??

Example: A company manufactures and sells two products. X sells for 10\$/unit and Y for 9\$/unit.
The cost of producing x units of product X and y units of product Y is
$$400 + 2x + 3y + (3x^2 + xy + 3y^2)\frac{1}{100}.$$

What is the local maximum profit?

Here ~~range~~ domain of (x,y) is unbounded, this would mean find local maximums ~~and~~ and compare.

Solution: Convert word problem $\rightarrow$ math.

Profit = Revenue − Cost.

Revenue $= 10x + 9y$.

$$\text{Profit} = \boxed{P(x,y)} = 10x + 9y - \left(400 + 2x + 3y + \frac{(3x^2 + xy + 3y^2)}{100}\right)$$

function we want to maximise.

$$= 8x + 6y - 400 - \frac{3}{100}x^2 - \frac{1}{100}xy - \frac{3}{100}y^2.$$

Check:

$$P_x = \frac{\partial P}{\partial x} = 8 - \frac{6}{100}x = \frac{1}{100}y.$$

$$P_y = \frac{\partial P}{\partial y} = 6 - \frac{6}{100}y - \frac{1}{100}x.$$

Solving for critical points.

$$P_x = 0 \Rightarrow 8 - \frac{6}{100}x - \frac{1}{100}y = 0 \quad \ldots (1)$$

and

$$P_y = 0 \Rightarrow 6 - \frac{6}{100}y - \frac{1}{100}x = 0 \quad \ldots (2)$$

By (1) $\quad 8 = \frac{6}{100}x + \frac{1}{100}y$

Multiplying by 100

$$800 = 6x + y.$$

$$\Rightarrow y = 800 - 6x$$

Replacing y in (2), we get,

$$6 - \frac{6}{100}(800 - 6x) - \frac{1}{100}(x) = 0.$$

Multiplying by 100

$$600 - 6(800 - 6x) - x = 0.$$

$$\Rightarrow 600 - 4800 + 36x - x = 0$$

$$\Rightarrow 35x = 4200$$

$\Rightarrow x = 120.$

Plugging into 1

$$8 - \frac{6}{100}(120) - \frac{1}{100}(y) = 0$$

$$\Rightarrow \quad 800 - 720 - y = 0$$

$$\Rightarrow \quad y = 80.$$

Critical point is $(120, 80)$.

Check: $P_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial x}\left(8 - \frac{6}{100}x - \frac{1}{100}y\right)$ ~~P_{xx} $P_{xx}(120,80)$~~

$$= -\frac{6}{100}$$

$$P_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial y}\left(8 - \frac{6}{100}x - \frac{1}{100}y\right)$$

$$= -\frac{1}{100}$$

$$P_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial y}\right) = \frac{\partial}{\partial y}\left(6 - \frac{6}{100}y - \frac{1}{100}x\right) = -\frac{6}{100}$$

Discriminant $D = P_{xx}P_{yy} - (P_{xy})^2$

$$= \left(-\frac{6}{100}\right)\left(-\frac{6}{100}\right) - \left(\frac{1}{100}\right)^2$$

Check. > 0.

$\therefore$ But $P_{xx}(120, 80) = -\frac{6}{100} < 0.$

Therefore $(120, 80)$ is a local maxima.

The maximum value is.

$$P(120, 80) = \cancel{400 + 2(120) + 3(80) + \left(3(120)^2 + (120)(80) + 3(80)^2\right)\frac{1}{100}}$$

$$= 8(120) + 6(80) - 400 - \frac{3}{100}(120)^2 - \frac{1}{100}(120)(80) - \cancel{\frac{3}{100}y^2} - \frac{3}{100}(80)^2$$