Midterm Review ①

Error Term in Simpson's Rule → General Request

• ~~Find express~~

Approximate $\int_0^{\pi} (\sin\theta + \cos\theta)\, d\theta$ by Simpson's Rule for $n = 4$ and find the error bound.

Solution: $S(4) = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$

$\Delta x = \frac{\pi - 0}{4} = \frac{\pi}{4}$; $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$.

$$\therefore S(4) = \frac{\pi}{12}\ \frac{\pi/4}{3}\Big[\sin(0) + \cos(0) + 4\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) + 2\left(\sin\left(\tfrac{\pi}{2}\right) + \cos\tfrac{\pi}{2}\right) + 4\left(\sin\tfrac{3\pi}{4} + \cos\left(\tfrac{3\pi}{4}\right)\right) + \sin(\pi) + \cos(\pi)\Big]$$

$$= \frac{\pi}{12}\left[\cancel{1} + 4\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) + 2 + 4\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + 0 - \cancel{1}\right]$$

$$= \frac{\pi}{12}\left[2 + 4\sqrt{2}\right]$$

$f(\theta) = (\sin\theta + \cos\theta)$; $f^{(4)}(\theta) = \cancel{\cos\theta - \sin\theta}\ \sin\theta + \cos\theta$

Now $|\sin\theta| < 1$, $|\cos\theta| < 1$

Thus. $|f^{(4)}(\theta)| \leq 1+1 = 2 \boxed{= k}$

The error is bounded by $\dfrac{k(\pi)^5}{180\, 4^{\cancel{4}}4} = \dfrac{2(\pi)^5}{(180)(4)^4}$

$\longrightarrow$ Ali Jatoi

1) $\dfrac{d}{dx}\left(\displaystyle\int_{x^2}^{e^{x^2}} (25t^{15} + t^{-15})\, dt \right)$

Suppose $\displaystyle\int \cancel{25t^{1}}\ \cancel{25}$
$f(t) = 25t^{15} + t^{-15}$
and its antiderivative is
$F(t)$.

Then by Fundamental Theorem of Calculus.

$$\int_{x^2}^{e^{x^2}} (25t^{15} + t^{-15})\, dt = F(e^{x^2}) - F(x^2)$$

Thus $$\frac{d}{dx}\left(\int_{x^2}^{e^{x^2}} (25t^{15} + t^{-15})\, dt \right) = F'(e^{x^2}) \frac{d}{dx}(e^{x^2}) - F'(x^2)\frac{d}{dx}(x^2)$$

$$= f(e^{x^2})(e^{x^2}\, 2x) - f(x^2)(2x)$$ [F is an antiderivative]

$$= (25(e^{x^2})^{15} + (e^{x^2})^{-15})(e^{x^2} 2x) - (25(x^2)^{15} + (x^2)^{-15})(2x)$$

~~A~~ Ali Jatoi

(3)

8) $\int \sin^{10}(x)\cos^3(x)\,dx$ $\longleftrightarrow$ Compare with $\int \sin^m(x)\cos^n x\,dx$

$m = 10$, $\boxed{n=3}$ odd.

$= \int \sin^{10}(x)\cos^2 x \cos x\,dx$

Substituting $\longrightarrow$ we get

$\sin(x) = u$

$du = \cos x\,dx$

$\cos^2 x = 1 - \sin^2 x$

$= 1 - u^2$

$\int u^{10}(1-u^2)\,du \longrightarrow$ Expression free of x. ~~and~~ ~~and using~~

$= \int u^{10}du - \int u^{12}du$

$= \frac{u^{11}}{11} - \frac{u^{13}}{13} + C$

$= \frac{(\sin x)^{11}}{11} - \frac{(\sin x)^{13}}{13} + C$ $\longleftarrow$ Do not forget to substitute back.

9) $\int \cot(x)\operatorname{cosec}^2 x\,dx$ $\longleftrightarrow$ Compare with.

$\int \cot^m(x)\operatorname{cosec}^n(x)\,dx$

$m = 1$ $n = 2$

$\downarrow$

odd.

$u = \operatorname{cosec} x$

$du = -\operatorname{cosec} x \cot(x)\,dx$

~~$u = \operatorname{cosec}(x)$~~

$= -\int u\,du$

$= -\frac{u^2}{2} + C$

$= -\frac{\operatorname{cosec}^2 x}{2} + C$

~~$\boxed{4 \& 6}$~~

~~Cannot~~ factori

$$\int \frac{dx}{x^2 + 5x + 6}$$

Factorise this

$$\int \frac{dx}{x^2 + \cancel{5x+6}\ 2x+3}$$

$\boxed{4 \& 6}$

Factorise

$$\int \frac{dx}{x^2 \pm 5x \pm 6}$$

Cannot factorise

$$\int \frac{dx}{x^2 + 2x + 3}$$

$$\int \frac{dx}{x^2+5x-6} = \int \frac{dx}{(x-2)(x-3)}$$

$$= A \int \frac{A}{}$$

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

$$\Rightarrow \quad 1 = A(x-3) + B(x-2)$$

$x = 2$ gives $1 = -A \Rightarrow A = -1$

$x = 3$ gives $B = 1$. $\therefore \dfrac{1}{(x-2)(x-3)} = -\dfrac{1}{x-2} + \dfrac{1}{x-3}$

$$\therefore \int \frac{dx}{x^2+5x-6} = -\int \frac{1}{x-2}\,dx + \int \frac{dx}{x-3}$$

$$= -\ln|x-2| + \ln|x-3| + C$$

~~What if it was~~ $\cancel{\int \frac{x^3\,dx}{x^2+5x-6}}$?

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$$\int \frac{dx}{x^2+2x+3} = \int \frac{dx}{x^2+2\cdot 1x+1^2+3-1^2}$$

$$= \int \frac{dx}{(x+1)^2+2}.$$

$x+1 = \sqrt{2}\tan\theta$

$dx = \sqrt{2}\sec^2\theta \, d\theta$

$\cancel{x = \sqrt{}}$

$\theta = \arctan\left(\frac{x+1}{\sqrt{2}}\right)$

$\sec^2\theta + 1 = \tan^2\theta$

$\cancel{(x+1) = \sqrt{2}\tan\theta}$

Substituting $\longrightarrow$

$$\int \frac{\cancel{dx}\ \sqrt{2}\sec^2\theta \, d\theta}{2(\tan^2\theta+1)}$$

$$= \frac{1}{\sqrt{2}}\int d\theta$$

$$= \frac{1}{\sqrt{2}}\theta + C$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C$$

Solve. $e^{t^2}y' = ty$ ~~withy~~ with $y(0)=1$.

$e^{t^2}y' = ty$

$\Rightarrow \cancel{\frac{y'}{y}} \cancel{=}$

$\Rightarrow e^{t^2}\frac{dy}{dt} = ty$

$\Rightarrow \int \frac{dy}{y} = \int t e^{-t^2} dt \longrightarrow$ Separted y and t and integrated.

$\Rightarrow \ln|y| = \int t e^{-t^2} dt$

Substituting we get, $\longrightarrow$ $-t^2 = u$
$\Rightarrow -2t\,dt = du$
$\Rightarrow t\,dt = -\frac{du}{2}$

$\ln|y| = -\int e^u \frac{du}{2}$

$= -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + c$

$= -\frac{1}{2}e^{-t^2} + c$

~~$\Rightarrow$ $\ln|y|$ $|y| = e^{\left(-\frac{1}{2}e^{-t^2}+c\right)}$~~

$\Rightarrow |y| = e^{\left(-\frac{1}{2}e^{-t^2}+c\right)} = e^{\left(-\frac{1}{2}e^{-t^2}\right)} \underbrace{e^c}_{\boxed{C' > 0}}$

$\Rightarrow |y| = C' e^{-\frac{1}{2}e^{-t^2}}$

$y(0)=1 \Rightarrow y>0 \Rightarrow |y| = y$ $\Rightarrow \boxed{\begin{array}{l} C' = e^{\frac{1}{2}} \\ y(t) = e^{\frac{1}{2}(1-e^{-t^2})} \end{array}}$

$\therefore \underbrace{\boxed{y(0)}}_{1} = C' e^{\left(-\frac{1}{2}e^{-(0)^2}\right)} = C' e^{\left(-\frac{1}{2}e^{0}\right)} = C' e^{-\frac{1}{2}}$