1st April, 2014

Recall:

- **Divergence test** $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$.
- **Integral test** $\sum_{n=1}^{\infty} a_n$ diverges/converges

  if and only if $\int f(x)\,dx$ converges.

  — $a_n = f(n)$, f - decreasing or increasing

  — $\boxed{\text{p-test}}$ $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges

  if and only if $p > 1$.
- **Telescopic series** $\sum_{k=1}^{\infty} \frac{1}{k(k+3)}$

  — partial fractions.
- **Geometric Series** $\sum_{k=1}^{\infty} a r^k$.

Squeeze Theorem (Sequences): $a_k \leq b_k \leq c_k$.

If $\lim_{k\to\infty} a_k = \lim_{k\to\infty} c_k = L$. Then $\boxed{\lim_{k\to\infty} b_k = L.}$

Comparison Theorem (Series): $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} c_k$ converges

then $\sum_{k=1}^{\infty} b_k$ converges as well.

Implications: • If $0 < a_k \le b_k$ and $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges.

• If $0 \le a_k \le b_k$ and $\sum_{k=1}^{\infty} a_k$ ~~converges~~ diverges then $\sum_{k=1}^{\infty} b_k$ diverges as well.

1) $\sum_{k=1}^{\infty} \frac{1}{k^3+1}$.

- Divergence test: - inconclusive.
- Integral test - hard to integrate.
- Comparison test. Note $k^3+1 > k^3$.

Then $0 < \frac{1}{k^3+1} < \frac{1}{k^3}$.

But $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by p-test for $p=3$.

$\therefore \sum_{k=1}^{\infty} \frac{1}{k^3+1}$ converges.

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2) ~~$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-1}$~~ $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}-1}$

- Divergence test - inconclusive.
- Integral test - hard to integrate.

But $\sqrt{k} - 1 < \sqrt{k}$.

$0 < \frac{1}{\sqrt{k}} < \frac{1}{\sqrt{k}-1}$, $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}}$ diverges ~~(By p-test $p=\frac{1}{2}$)~~

by p-test for $p = \frac{1}{2}$.

Thus $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k-1}}$ diverges as well.

Limit Comparison Test:

~~Supp~~ ~~If $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ are positive and~~

If a_k, b_k are positive and $\lim_{k\to\infty} \frac{a_k}{b_k} = L$.

(1) If $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ ~~conv~~; $\sum_{k=1}^{\infty} b_k$ either both converges or both diverges.

(2). $L = 0$. $\sum b_k$ converges then $\sum a_k$ converges.

(3) $L = \infty$ $\sum b_k$ diverges then $\sum a_k$ diverges.

Example: $\sum_{k=1}^{\infty} \cancel{\frac{\ln k}{k^2}} \frac{k^2 - k + 6}{k^3 + k} = \cancel{\sum_{k=1}^{\infty}} a_k$.

~~Compare with~~ $\cancel{\frac{1}{k^3}}$

Highest term in numerator k^2

Highest term in denominator k^3.

Then compare with $\cancel{\sum \frac{k^2}{k^3}}$ $\sum_{k=1}^{\infty} \frac{k^2}{k^3} = \frac{1}{k} = \sum_{k=1}^{\infty} b_k$.

$\cancel{\lim_{k\to\infty} \frac{\frac{k^2-k+6}{k^3+k}}{\frac{1}{k}}}$ $\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{k^2-k+6}{k^3+k}}{\frac{1}{k}}$

$= \lim_{k\to\infty} \frac{k(k^2-k+6)}{k^3+k}$.

4.

$$\cancel{= \lim_{k\to\infty} \frac{k(k^2 - k + 6)}{k(k^2+1)} = \lim_{k\to\infty}}$$

$$= \lim_{k\to\infty} \frac{k^3 - k^2 + 6k}{k^3 + k} = \lim_{k\to\infty} \frac{\frac{k^3 - k^2 + 6k}{k^3}}{\frac{k^3+k}{k^3}}$$

[Divide by highest power of k]

$$= \lim_{k\to\infty} \frac{\frac{k^3}{k^3} - \frac{k^2}{k^3} + 6\frac{k}{k^3}}{\frac{k^3}{k^3} + \frac{k}{k^3}} = \lim_{k\to\infty} \frac{1 - \frac{1}{k} + \frac{6}{k^2}}{1 + \frac{1}{k^3}}$$

$$= \frac{1}{1} = 1$$

$$\boxed{L = 1}$$

~~But~~ (Part ① of limit comparison Test)

And $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

$\therefore \sum_{k=1}^{\infty} \frac{k^2 - k + 6}{k^3 + k}$ diverges as well.

$\cancel{\sum_1 \frac{\ln k}{k^3}}$ $\sum_{k=1}^{\infty} \frac{\ln k}{k^2}$ ~~cannot apply integral test not decreasing.~~

Compare with $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ (or integral test with justification)

Ratio Test ~~Test~~

$\sum_{k=1}^{\infty} a_k.$ ~~Let $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.~~

Let $r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}.$

1) If $0 \le r < 1$ the series converges.
2) If $r > 1$ (including $r = \infty$), the series diverges.
3) If $r = 1$, the test is inconclusive.

1) $\sum_{k=1}^{\infty} \frac{1}{k^2}$ $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{1/k^2} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2}$

$= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 = \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}}\right)^2 = 1$

inconclusive

(Use integral test).

2) Geometric Series

$\sum_{k=1}^{\infty} \frac{1}{r^k}$. $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{r^{k+1}}}{\frac{1}{r^k}} = \lim_{k\to\infty} \frac{r^k}{r^{k+1}}$

$= \lim_{k\to\infty} \frac{1}{r} = \frac{1}{r}$. Thus converges is $r > 1$,

diverges if $r < 1$

inconclusive if $r = 1$

[Use divergence test] for $r = 1$.]

3 $\displaystyle\sum_{k=1}^{\infty} \frac{x^p}{p!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{x^{p+1}}{p+1!}}{\frac{x^p}{p!}} = \lim x^p$$

3 $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{x^k}{k!}$ $\quad \displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}$

$$= \lim_{n\to\infty} \frac{x^{n+1}}{x^n} \cdot \frac{\cancel{(n+1)(n)(n-1)\cdots 1}}{\cancel{n(n-1)\cdots 1}} \quad n(n+$$

$$= \lim_{n\to\infty}$$

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3 $\displaystyle\sum_{k=1}^{\infty} x^k/k!$

$$\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \lim_{n\to\infty} \frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} x \frac{n(n-1)\cdots 1}{(n+1)\cancel{n}\cancel{(n-1)}\cdots 1} = x \lim_{n\to\infty} \frac{1}{(n+1)} = 0.$$

$\therefore \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k!}$ converges always. for all values of x

Strategy: Given a series, what should we do?

1) Divergence test

2) Identify telescopic / geometric / p-series.

3) ~~Can yo~~ Increasing / decreasing - Can you integrate? - Integral test

4) If you see k^k, $k!$, a^k maybe use the ratio test

5) Similar to something ~~which you~~ in $\boxed{2}$, then use limit comparison test / Comparison test.

Taylor Series

Suppose $|x| < 1$ Then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Note: ~~function~~. Let $f(x) = \frac{1}{1-x}$.

$f(0) = 1$ $\qquad$ $f(0) = 1$

$f'(0) = \left.\frac{1}{(1-x)^2}\right|_{x=0} = 1$ $\qquad$ $f'(0) = 1$

$f''(0) = \left.\frac{2}{(1-x)^3}\right|_{x=0} = 2$. $\qquad$ $\frac{f''(0)}{2!} = 1$.

$f'''(0) = \left.\frac{3 \cdot 2}{(1-x)^4}\right|_{x=0} = 3 \cdot 2 = 3!$ $\qquad$ $\frac{f'''(0)}{3!} = 1$.

Check $\frac{f^{(n)}(0)}{n!} = 1$

Thus $f(x) = \sum_{k=0}^{\infty} \boxed{\frac{f^{(k)}(0)}{k!}} x^k$. for $|x| < 1$.

Power Series: ~~Maclaurin Series~~

~~Taylor series~~. Taylor Series

Taylor Series for f centred at a is ~~2nd Term~~

$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$

~~1st Term~~ ~~3rd term~~

$= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum a_k$

a_k depends on x.

$a_k = \frac{f^{(k)}(a)(x-a)^k}{k!}$ $\boxed{0! = 1! = 1}$

~~kth term~~

Maclaurin Series

Taylor Series with $a = 0$.

~~e^x about~~ Find ~~Taylor~~ Maclaurin series of $f(x) = e^{x}$ ~~about 0~~.

$f(0) = 1$

$f'(0) = e^0 = 1$

$f''(0) = 1$.

$f^{(n)}(0) = 1$ for all n.

Then the taylor series is

$$\sum_{k=0}^{\infty} \frac{1 \cdot x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$