Applications of Integration (Continued)

Question.

$1 = 1$

$1 + \frac{1}{2} = \frac{3}{2}$

$1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{137}{60}$

What is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

$= \sum_{i=1}^{\infty} \frac{1}{i}$? Does this make any sense?

$1 = 1$

$1 + \frac{1}{2} = \frac{3}{2} = 1.5$

$1 + \frac{1}{2} + \frac{1}{2^2} = \frac{7}{4} = 1.75$

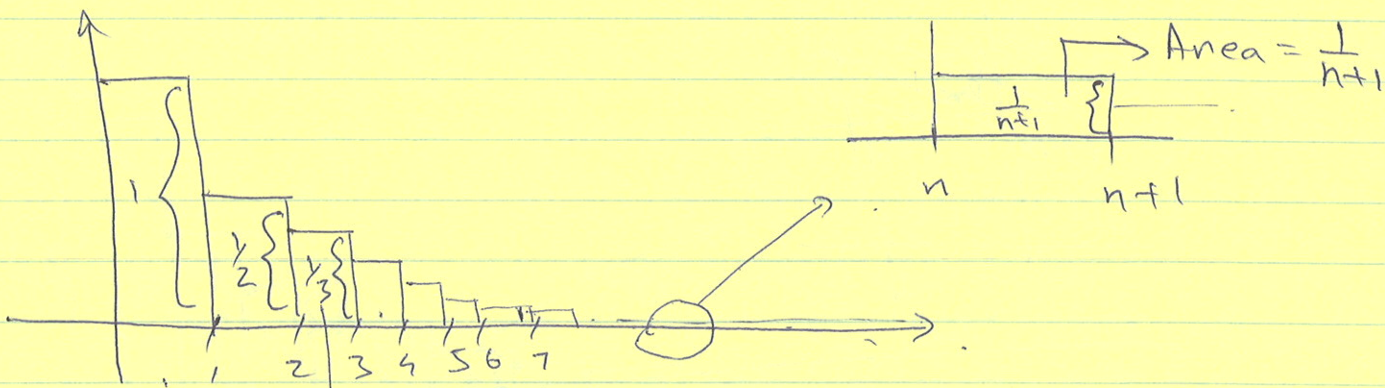
$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} = \frac{15}{8} = 1.875$

$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} = 1.9375$

What is $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots$

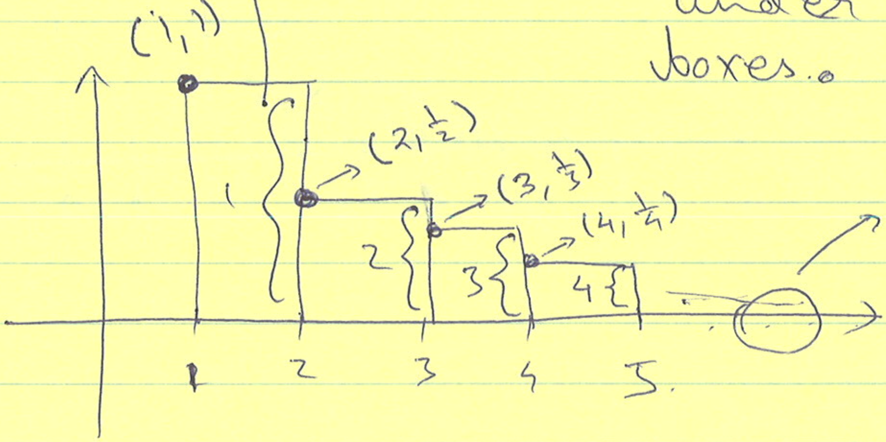
$= \sum_{i=0}^{\infty} \frac{1}{2^i}$? Does ~~the~~ this even make sense?

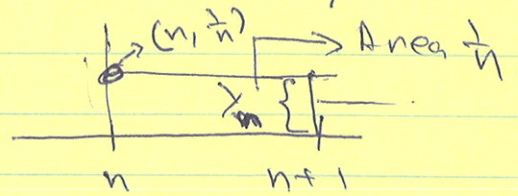
How is this related to integrals?



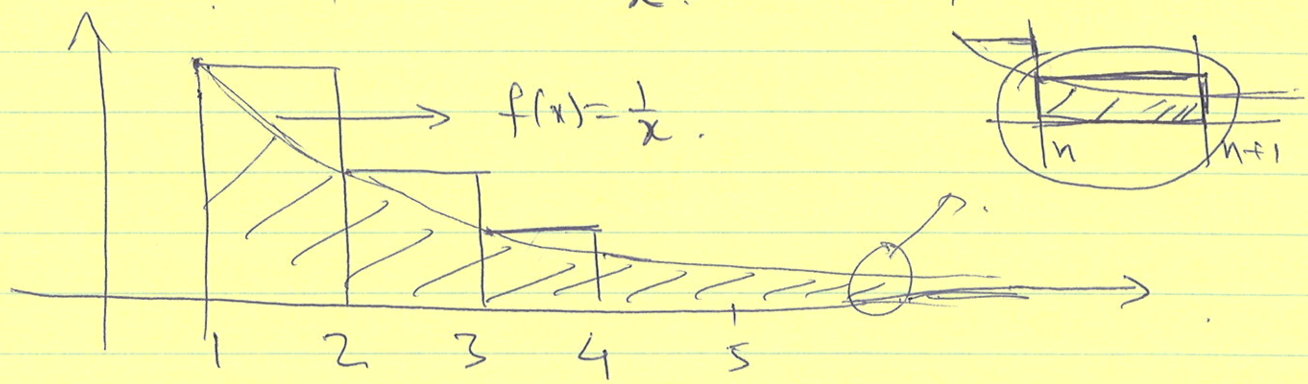
Question: $\sum_{i=1}^{\infty} \frac{1}{i}$ finite is

equivalent to finite area under the curve / in the boxes.





Consider function $f(x)=\frac{1}{x}$ $\longrightarrow$ $(x, f(x)) = (x, \frac{1}{x})$.



$\frac{1}{n}$ = Notice area of ~~rectan~~ n^{th} rectangle

$> \int_n^{n+1} \frac{1}{x}\,dx$.

Thus $\displaystyle\int_1^n f(x)\,dx < \sum_{i=1}^{n-1} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n-1}$

$\parallel$

$\displaystyle\int_1^n \frac{1}{x}\,dx$

$\parallel$

$\{\ln|x|\}\Big|_1^n$

$= \cancel{\ln n}\ \ln(n) - \ln(1)$

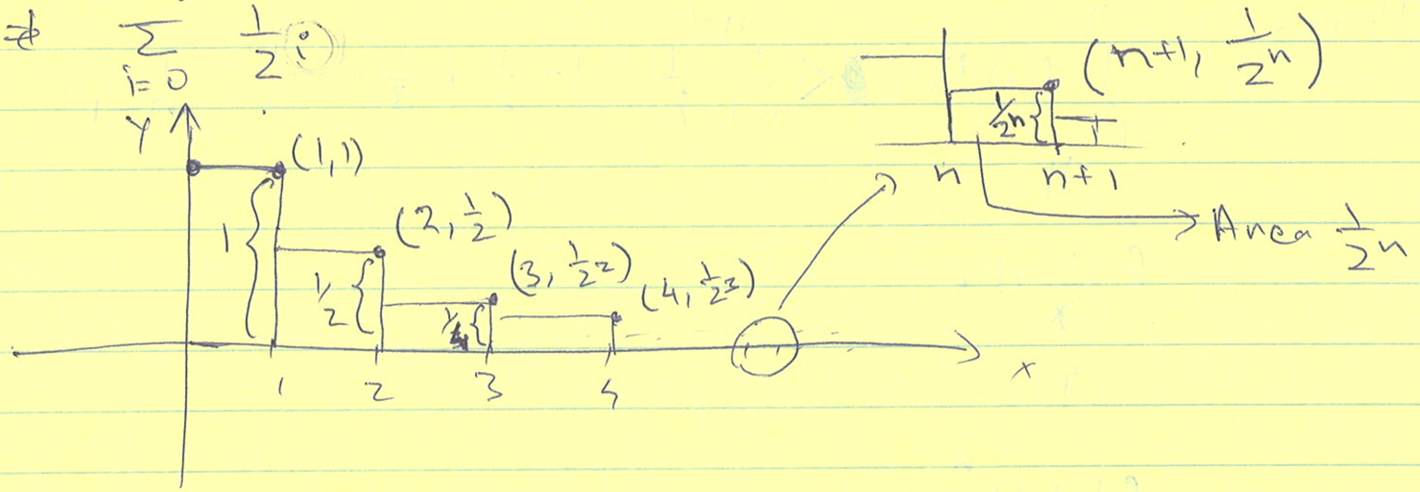
$= \ln(n).$

~~The~~ But ~~lim~~ $\ln(n)$ diverges as $n \to \infty$

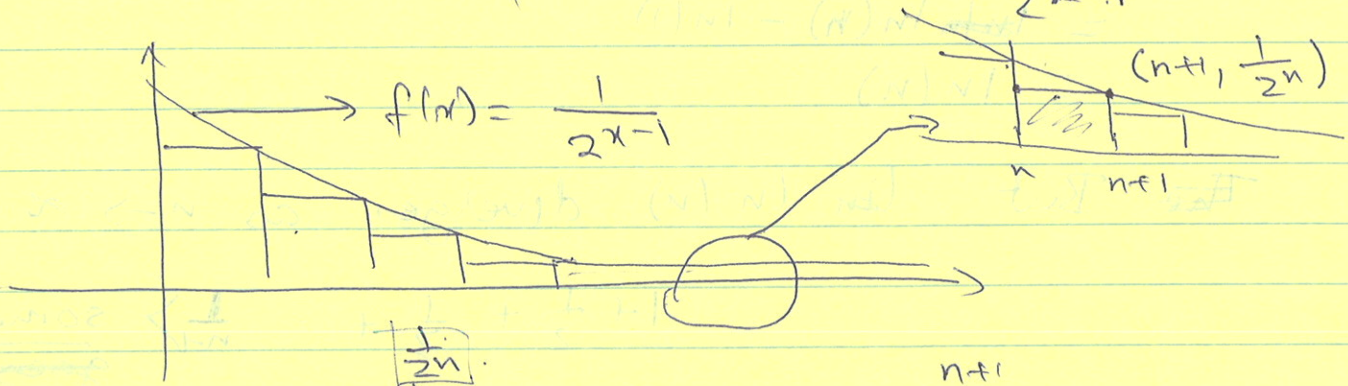
$1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n-1} >$ something ~~goes~~ which is going to infinity

$\therefore \displaystyle\sum_{i=1}^{\infty} \frac{1}{i}$ diverges. (infinite area under rectangles).

1st $\sum_{i=0}^{\infty} \frac{1}{2^i}$



Consider the function $f(x) = \frac{1}{2^{x-1}}$



1 · Area of n^{th} rectangle $< \int_n^{n+1} \frac{1}{2^{x-1}}\, dx.$

$$\therefore \int_0^n f(x)\, dx > 1 + \frac{1}{2} + \frac{1}{2^2} \quad + \frac{1}{2^{n-1}}$$

$$||$$

$$\int_0^n \frac{1}{2^{x-1}}\, dx.$$

$$= \int_0^n 2^{-x+1}\, dx = 2 \int_0^n 2^{-x}\, dx.$$

$2^{-x} = y$

Then $\log 2^{-x} = \log y$.

$\Rightarrow -x \log 2 = \log y$.

$\Rightarrow \log y = -x \log 2$.

$\Rightarrow e^{\log y} = e^{-x \log 2}$

$\Rightarrow 2^{-x} = y = e^{-x\log 2} = e^{(-\log 2)x}$. — — (*)

$$\therefore \int_0^n 2^{-x}\,dx = \int_0^n e^{(-\log 2)x}\,dx \quad [\text{By } *]$$

$$= \left.\frac{e^{(-\log 2)x}}{(-\log 2)}\right|_0^n$$

$$= \left.\frac{2^{-x}}{(-\log 2)}\right|_0^n \quad [\text{By } * \text{ Again}].$$

$$= \frac{2^{-n} - 2^{-0}}{-\log 2} = \frac{2^{-n} - 1}{-\log 2}$$

$$= \frac{1 - 2^{-n}}{\log 2}.$$

$$\text{i.e. } \int_0^\infty 2^{-x}\,dx = \lim_{n\to\infty} \int_0^n 2^{-x}\,dx = \lim_{n\to\infty} \frac{1 - 2^{-n}}{\log 2}.$$

$$= \frac{1 - \lim_{n\to\infty} 2^{-n}}{\log 2} = \frac{1-0}{\log 2} = \frac{1}{\log 2}.$$

$n \to \infty$

$\cancel{n \to \infty}$

$\cancel{2^{-n} \to 0}$

$2^{-n} = \frac{1}{2^n}$

$2^n \longrightarrow \infty$

$\frac{1}{2^n} \longrightarrow 0$

But $\lim_{n\to\infty} \int_0^n 2^{-x}\,dx > \lim_{n\to\infty} \frac{1}{2} + \cdots \frac{1}{2^{n-1}} > 0$

$\downarrow$ Converges

$\parallel$ $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{1}{2^i}$

Therefore. $\sum_{i=0}^{n-1} \frac{1}{2^i}$ converges as well.

$\boxed{\text{Integral Test}}$

Theorem: Suppose f is a continuous, positive decreasing function for $x \geq 1$ and let $a_K = f(k)$ for $k = 1, 2, \ldots$ Then

$$\sum_{K=1}^{\infty} a_K \text{ and } \int_1^{\infty} f(x)\,dx.$$

either both converge or both diverge.

p- series

For what value of $p > 0$ does $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converge?

We checked $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

So let $p \neq 1$.

$$\sum_{k=1}^{\infty} \underbrace{\frac{1}{k^p}}_{= a_k} \rightsquigarrow \text{Zeta function}.$$

If $f(x) = \frac{1}{x^p}$ then $f(k) = \frac{1}{k^p}$.

~~f is decreasing~~

f is a decreasing positive function. when $p > 0$.

~~We~~. So ~~$\int_{k=1}^{\infty} \frac{1}{k^p}$~~ $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges if and only if $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{c \to \infty} \int_1^{c} \frac{1}{x^p}\,dx = \lim_{c \to \infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^c$$

$$= \lim_{c \to \infty} \frac{c^{-p+1} - 1^{-p+1}}{1-p}$$

$$= \lim_{c \to \infty} \frac{1 - c^{1-p}}{p-1}$$

$$\lim_{c \to \infty} c = \infty$$

$$\lim_{c \to \infty} \frac{1}{c} = 0.$$

$$\cancel{c^{-p+1} = \frac{c}{c^{p}} \quad c \cdot c^{-p} = \frac{c}{c^{p}} \quad \frac{c}{c^{p}}}$$

$$\cancel{\lim_{c \to \infty} c^{-p+1}}$$

~~If $p > 1$ Then $1-p$~~

~~Then $0 >$~~

$$\cancel{\text{Then} \lim_{c \to \infty} c^{-p+1} = \lim_{c \to \infty} \frac{1}{c^{-(-p+1)}}}$$

~~$-p+1 < 0$~~

~~$\Rightarrow -(-p+1) > 0$~~

~~$c \longrightarrow \infty$~~

~~$c^{-(p+1)} \longrightarrow \infty$~~

~~$\frac{1}{c^{-(-p+1)}} \longrightarrow 0$~~

If $1-p > 0$ (or $p < 1$)

$$\lim_{c \to \infty} c^{1-p} = \infty \quad \therefore \int_1^\infty \frac{1}{x^p}\,dx \text{ diverges}$$

If $(1-p) < 0$ (or $p > 1$)

$$\lim_{c \to \infty} c^{1-p} = \lim_{c \to \infty} \frac{1}{c^{-(1-p)}} = 0 \quad \text{converges}$$

$1-p < 0$

$-(1-p) > 0$

$c^{-(1-p)} \longrightarrow \infty$

$\boxed{Ex}$ ~~Find~~

1) $$\frac{d}{dx}\left(\lim_{n \to \infty} \left[\left(1 + \left(1+2\frac{x}{n}\right)^3 + \left(1+2\frac{2x}{n}\right)^3 + \left(1+2\cdot\frac{3x}{n}\right)^3 + \cdots + \left(1+2\frac{(n-1)x}{n}\right)^3 \right) \frac{x}{n} \right] \right)$$

2) A ~~firm~~ firm makes x units of keys and y units of locks. For some strange reason

$$x^2 + 10y^2 = 50 \quad \text{and } x \geq 0, y \geq 0.$$

Selling a key bring \$1 profit and selling a lock brings \$2 profit.

Find x and y to maximise profit.

3) ~~Ex~~ ~~Ex~~ ~~Find~~. Let $f(x) = \begin{cases} \dfrac{k|x|}{1+x^2} & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

For what value of of k. is f a pdf.

Find $\mathbb{E}(X)$

(4) Is $\sum_{k=1}^{\infty} \frac{k^2}{1+k^3}$ convergent or divergent?