ABSOLUTE MAXIMAS AND MINIMAS

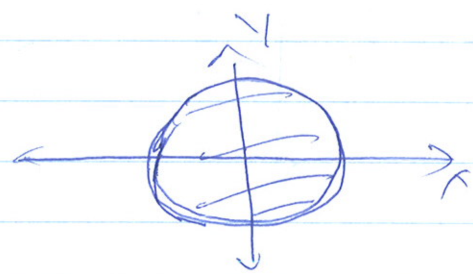
21st January.

We developed techniques for finding the local maximums and minimums in the last class. Absolute maximum and minimum can be found on Closed and Bounded regions.

Closed and Bounded regions

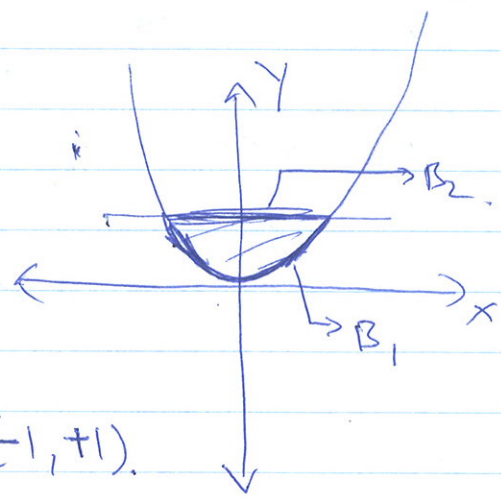
Closed sets sets which include their boundary.

What is a boundary?



- Boundary of $x^2+y^2 \leq 1$

  is $x^2+y^2=1$



- Boundary of $x^2 \leq y$, $y \leq 1$

  has two parts: $x^2=y$ and $y=1$

~~For Rooting~~

They intersect at $(1,1)$ and $(-1,+1)$.

~~(Plug $y=1$ into $x^2=y$~~

The boundary is $B_1 \cup B_2$ where

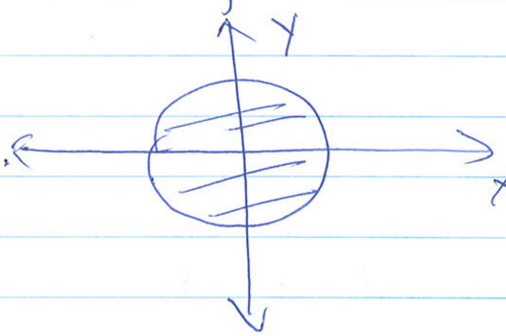
B_1 is (x,y) such that $x^2 \leq y$ and $-1 \leq x \leq 1$

B_2 is (x,y) such that $y=1$ and $-1 \leq x \leq 1$

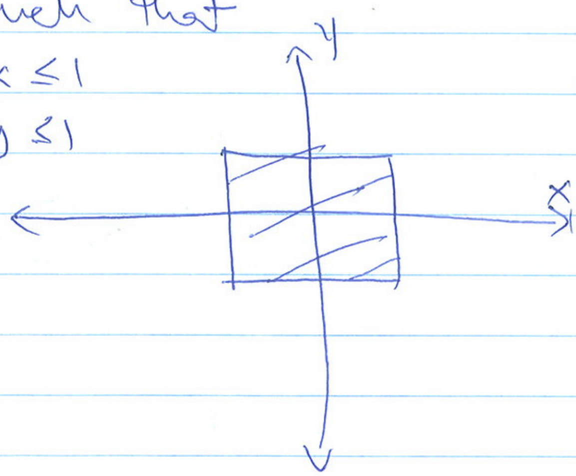
Bounded sets are sets which ~~as~~ can be enclosed in a big enough circle.

Closed and Bounded.

(x,y) such that
$x^2+y^2 \leq 1$

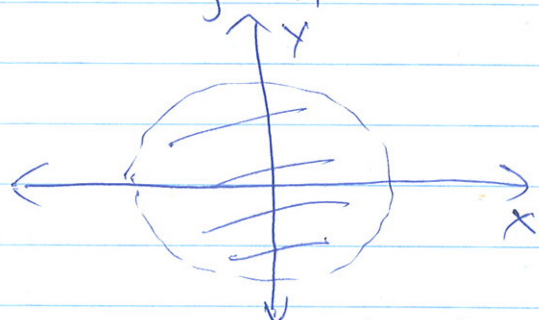


(x,y) such that
$-1 \leq x \leq 1$
$-1 \leq y \leq 1$



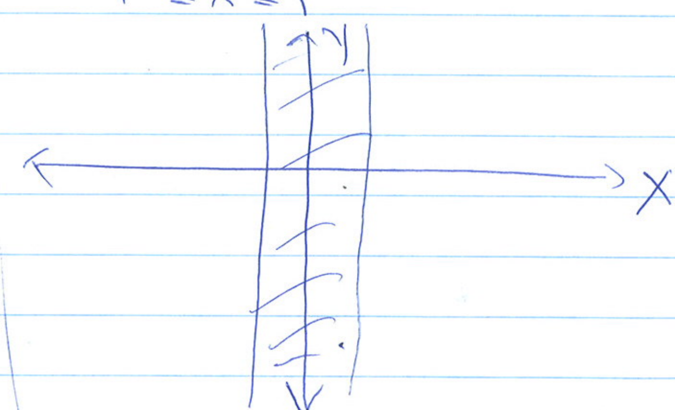
Not Closed or Not Bounded.

(x,y) such that
$x^2+y^2 < 1$



not closed.

(x,y) such that
$-1 \leq x \leq 1$



not bounded

f has an <u>absolute maximum</u> at (a,b) if

$f(a,b) \geq f(x,y)$ for all (x,y) in the domain of f.

f has an <u>absolute minimum</u> at (a,b) if

$f(a,b) \leq f(x,y)$ for all (x,y) in the domain of f.

- Finding absolute maximum / minimum on the closed and bounded set R.

(A) Find all critical values in the set R.

(B) ~~Draw~~ Sketch R and find the boundary.

(C) Find the maximum and minimum values on the boundary.

(D) Find the greatest and smallest among (A) and (C).

<u>Example</u>: Find the absolute maximum ~~and~~ ~~maximum~~ minimum of $f(x,y) = x^2 + y^2 - x - y$ on the region R satisfying

$x^2 \leq y$ and $y \leq 1$.

Solution: Step A.

$f_x = 2x - 1$
$f_y = 2y - 1$

$f_x = 0$
$\Rightarrow 2x - 1 = 0$
$\Rightarrow x = \frac{1}{2}$.

$f_y = 0$
$\Rightarrow 2y - 1 = 0$
$\Rightarrow y = \frac{1}{2}$.

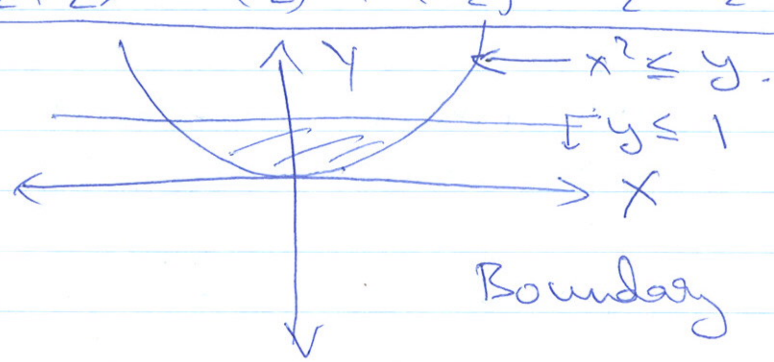
The critical point is $(\frac{1}{2}, \frac{1}{2})$.

and it satisfies $x^2 \leq y$ and $y \leq 1$

So it lies in the domain.

$$\boxed{f(\tfrac{1}{2}, \tfrac{1}{2}) = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 - \tfrac{1}{2} - \tfrac{1}{2} = -\tfrac{1}{2}}$$

Step B



Boundary $B_1 : x^2 = y \quad -1 \leq x \leq 1$
$B_2 : y = 1 \quad -1 \leq x \leq 1$

Step C Parametrisation can be important

$\boxed{\text{On } B_1}$ $x^2 = y$ and $-1 \leq x \leq 1$

expressing in terms of x.

Let $g(x) = f(x, y) = f(x, x^2)$

$$= x^2 + (x^2)^2 - x - x^2 = x^4 - x.$$

Critical point: $g'(x) = 4x^3 - 1 = 0$

$$\Rightarrow x^3 = \frac{1}{4}$$

$$\Rightarrow x = \left(\frac{1}{4}\right)^{1/3}.$$

$$y = x^2 = \left(\left(\frac{1}{4}\right)^{1/3}\right)^2 = \left(\frac{1}{4}\right)^{2/3}.$$

$$g\left(\left(\tfrac{1}{4}\right)^{1/3}\right) = f\left(\left(\tfrac{1}{4}\right)^{1/3}, \left(\tfrac{1}{4}\right)^{2/3}\right) = \left(\left(\tfrac{1}{4}\right)^{1/3}\right)^4 - \left(\tfrac{1}{4}\right)^{1/3}$$

$$= \boxed{\left(\tfrac{1}{4}\right)^{4/3} - \left(\tfrac{1}{4}\right)^{1/3}}$$

$$= \left(\tfrac{1}{4}\right)^{1/3}\left(\tfrac{1}{4} - 1\right)$$

End Points: $x = -1$ and $x = 1$. $= -\left(\frac{1}{4}\right)^{1/3}\frac{3}{4}.$

$x = -1$, $y = x^2 = 1$

$$g(-1) = f(-1, 1) = (-1)^4 - (-1) = 1 + 1 = \boxed{2}.$$

$x = 1$ $\quad y = x^2 = 1$

$$g(1) = f(1, 1) = 1 - 1 = \boxed{0}.$$

$\boxed{\text{On } B_2}$ $\quad y = 1 \quad -1 \le x \le 1$

$$h(x) = f(x, 1) = x^2 + 1 - x - 1 = x^2 - x$$

Critical point: $h'(x) = 2x - 1 = 0$

$$\Rightarrow x = 1/2$$

~~$y = x^2 = \frac{1}{4}$~~.

$$h(\tfrac{1}{2}) = f(\tfrac{1}{2}, 1) = (\tfrac{1}{2})^2 - (\tfrac{1}{2}) = \boxed{-\tfrac{1}{4}}.$$

<u>End Points</u>. $x = -1$, $x = 1$.

$x = -1$, $y = (-1)^2 = 1$.

$$h(-1) = f(-1, 1) = (-1)^2 - (-1) = \boxed{2}.$$

$x = 1$ $y = (1)^2 = 1$

$$h(1) = f(1, 1) = (1)^2 - 1 = \boxed{0}.$$

Absolute Maximum is at $(-1, 1)$ value is 2.
Absolute Minimum is at $(\tfrac{1}{2}, \tfrac{1}{2})$ value is $-\tfrac{1}{2}$.

<u>Example</u>: Find the maximum value of the function. $f(x,y) = x^2 + y^2 - xy$ on

$$R = \{ x^2 + y^2 \leq 1 \}.$$

Solution: Critical points: $f_x = 2x - y = 0 \Rightarrow 2x = y$ --- (1).

$f_y = 2y - x = 0 \Rightarrow 2y = x$ --- (2)

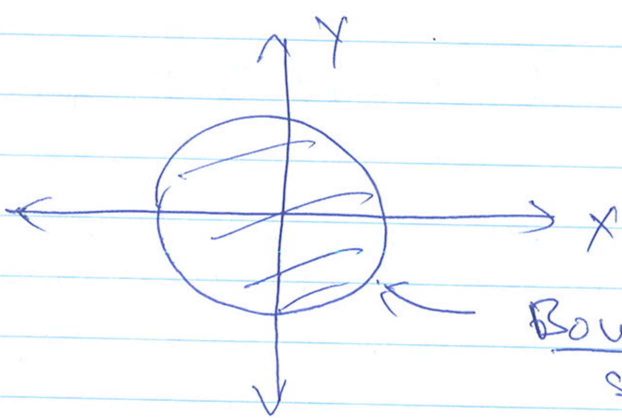
Plugging (1) into (2) we get.

$4x = x \Rightarrow x = 0. \Rightarrow y = 0.$

Critical point $(0,0)$

Critical Value $= f(0,0) = \boxed{0}$.

Step B



Boundary: (x,y) such that $x^2+y^2 = 1$

Step (C). ~~(x, y) su Parametrisation~~

Parametrisation:

(x,y) such that $x^2+y^2=1$ is the same as

$(\sin\theta, \cos\theta)$ when $0 \leq \theta \leq 2\pi$.

$$g(\theta) = f(\sin\theta, \cos\theta)$$

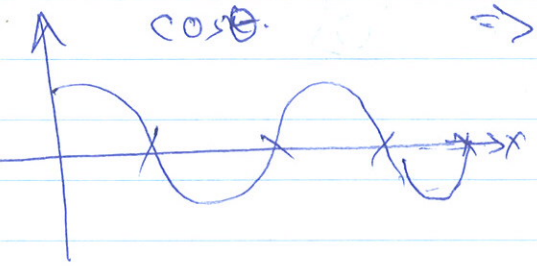
$$= \sin^2\theta + \cos^2\theta - \sin\theta\cos\theta$$

$$= 1 - \sin\theta\cos\theta$$

$$= 1 - \frac{1}{2}2\sin\theta\cos\theta = 1 - \frac{1}{2}\sin 2\theta.$$

~~Critical~~

Critical Point $g'(\theta) = \frac{1}{2}2\cdot\cos 2\theta = \cos 2\theta = 0.$



$$\Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}.$$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

$$g\left(\frac{5\pi}{4}\right) = g\left(\frac{\pi}{4}\right) = f\left(\sin\left(\frac{5\pi}{4}\right), \cos\left(\frac{5\pi}{4}\right)\right) = 1 - \frac{1}{2}\sin\left(2\cdot\frac{\pi}{4}\right)$$

$$= 1 - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)$$

$$= 1 - \frac{1}{2}(1) = \boxed{\frac{1}{2}}$$

$$g\left(\frac{3\pi}{4}\right) = g\left(\frac{7\pi}{4}\right) = f\left(\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right)\right)$$

$$= 1 - \frac{1}{2}\sin\left(2\cdot\left(\frac{3\pi}{4}\right)\right)$$

$$= 1 - \frac{1}{2}\sin\left(\frac{3\pi}{2}\right).$$

$$= 1 - \frac{1}{2}(-1) = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$$

End Points 'Circle has no' ends

(9)

Step

$\boxed{D}$

The maximum is at $\left(\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right)\right)$

and $\left(\sin\left(\frac{7\pi}{4}\right), \cos\left(\frac{7\pi}{4}\right)\right)$

and the maximum value is $3/2$.

Is there a better way?