www.math.ubc.ca/~nishant/Teaching/m105.html

Method of Lagrange Multipliers.

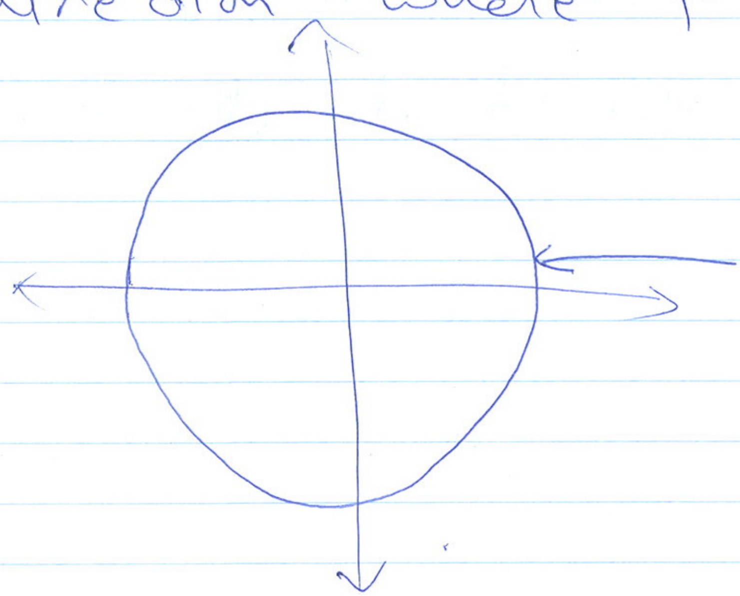
Task 1 → google list of thing named after Lagrange

Suppose $f(x,y)$ is given.

We can find f_x and f_y, the partial derivatives. What does

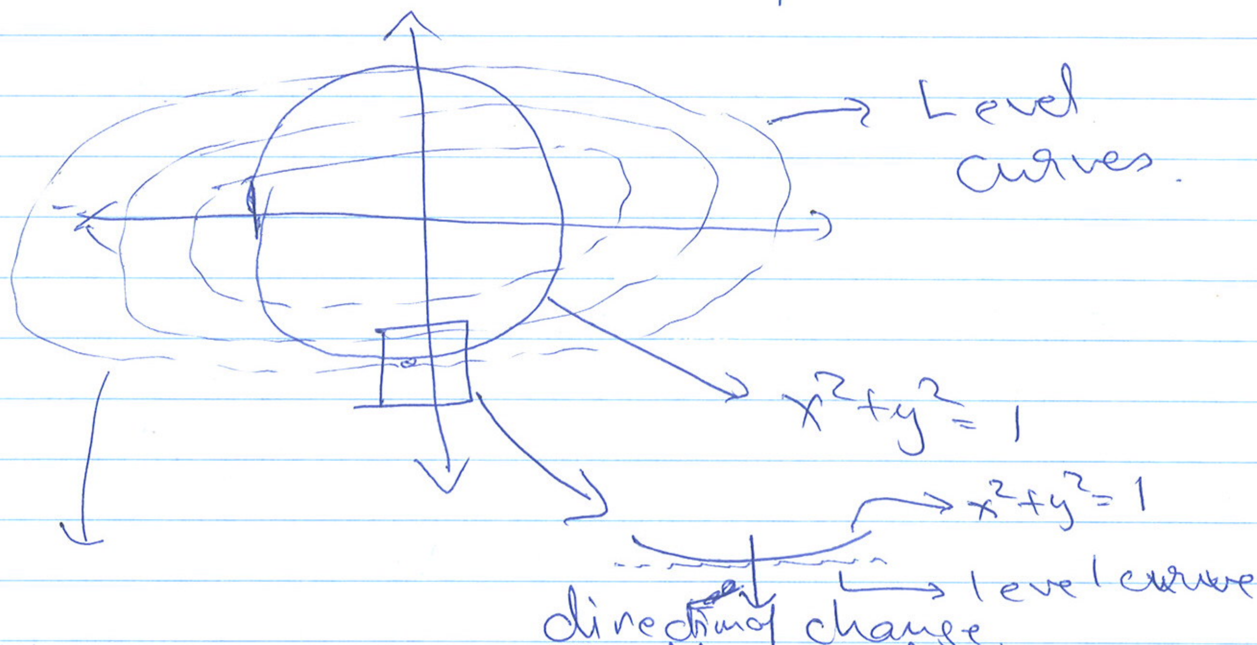
$\boxed{\nabla f} = \langle f_x(x,y), f_y(x,y) \rangle$ represent?

This vector is called the gradient vector of the function f. It indicates the direction where f is changing.



Suppose we want to maximise f on the circle $x^2+y^2=1$

Draw the level curves of f.



Maximum is attained where the highest level curve intersects the circle. So at this point the gradient of both f and x^2+y^2 should be the same.

The Method: Maximise/Minimise $f(x,y)$ on the curve $g(x,y)=0$.

$f(x,y)$ is called the objective function

$g(x,y)$ is called the constraint equation.

Assume $\langle g_x(x,y), g_y(x,y)\rangle = \langle 0,0\rangle$ on the curve $g(x,y)=0$.

~~To~~

Step1> Find x, y, λ that satisfy the equation.

$$\left(\begin{array}{c} f_x(x,y) = \lambda\, g_x(x,y) \\ f_y(x,y) = \lambda\, g_y(x,y) \\ g(x,y)=0 \end{array}\right) \quad \cdots\ (*)$$

Step2> Evaluate f at all points (x,y) that come from step 1 and find the maximum/minimum.

Note: (*) is ~~se~~ asking for ~~is~~ ∇f parallel to be ∇g.

~~or~~ and ~~is~~ the constraint to be satisfied.

Example from last class

Find the maximum value of the function

$$f(x,y) = x^2 + y^2 - xy \text{ ~~on~~ on}$$

Utility

$$x^2 + y^2 = 1 \longrightarrow \text{constraint equation}$$

$$g(x,y) = x^2 + y^2 - 1 = 0.$$

~~Using Lagrange Multipliers.~~

- ~~First equation:~~

$$\cancel{f_x(x,y) = \lambda g(x,y)} \text{ ~~yields~~}$$

$$\|$$

$$\cancel{2x - y}$$

$$f_x(x,y) = \cancel{2y - x}\ 2x - y. \qquad g_x(x,y) = 2x$$

$$f_y(x,y) = 2y - x \qquad g_y(x,y) = 2y.$$

$f_x(x,y) = \lambda g_x(x,y)$ yields

$2x - y = \lambda 2x$ --- (1)

$f_y(x,y) = \lambda g_y(x,y)$ yields

$2y - x = \lambda 2y$ --- (2)

$g(x,y) = 0$

$\Rightarrow x^2 + y^2 - 1 = 0$ --- (3)

Solve for x, y, λ.

Hint: Isolate λ first.

From (1) $\Rightarrow \frac{2x - y}{2x} = \lambda$

Eliminate λ next.

Plugging λ in (2)

$$(2y - x) = \left(\frac{2x-y}{\cancel{2}x}\right)\cancel{2}y$$

Multiplying by x, we get,

$$x(2y - x) = (2x - y)\,y$$

$$\Rightarrow \cancel{2xy} - x^2 = \cancel{2xy} - y^2$$

$$\Rightarrow x^2 = y^2$$

Use in 3.

$$x^2 + y^2 = 1$$

$$\Rightarrow 2x^2 = 1$$

$$\Rightarrow x^2 = \frac{1}{2}$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

$$\Rightarrow y = \pm \frac{1}{\sqrt{2}}$$

The points found are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Step 2: Plug ~~it in~~ the value in. Minimum.

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}$$

$$f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \boxed{\frac{3}{2}}$$

maximum

Consumer Behavior

$$U = f(l, g)$$

utility — happiness

leisure — less assignments

goods — ~~mark~~ term grades

Assumption: u increases if either any of l or g increase.

~~Assumption~~: level curves $\longrightarrow$ curves of indifference.

~~Rati~~ ~~Usu~~ linear relation between l and g.

Specific model $\rightarrow$ Cobb Douglas Model.

Example: Maximise $u = f(l, g) = l^{4/5} g^{1/5}$

with constraint $5l + 4g = 20$

not bounded

$$f_l = \frac{4}{5} l^{\frac{4}{5}-1} g^{\frac{1}{5}} = \frac{4}{5} l^{-\frac{1}{5}} g^{\frac{1}{5}}$$

$$f_g = \frac{1}{5} l^{\frac{4}{5}} g^{\frac{1}{5}-1} = \frac{1}{5} l^{4/5} g^{-4/5}$$

$$h_l = \frac{\partial}{\partial l}(5l + 4g - 20) = 5$$

$$h_g = \frac{\partial}{\partial g}(5l + 4g - 20) = 4$$

Equations: $f_l = \lambda h_l$

$$\Rightarrow \frac{4}{5} l^{-\frac{1}{5}} g^{\frac{1}{5}} = \lambda 5 = 5\lambda \quad \cdots (1)$$

$$f_g = \lambda h_g$$

$$\frac{1}{5} l^{\frac{4}{5}} g^{-\frac{4}{5}} = \lambda . 4 = 4\lambda \quad \cdots (2)$$

Constraint $h(l, g) = 0$

$$\Rightarrow 5l + 4g - 20 = 0 \quad \cdots (3)$$

By (1) $\lambda = \frac{4}{25} l^{-\frac{1}{5}} g^{\frac{1}{5}}$

Plugging into (2) $\frac{1}{\cancel{5}} l^{4/5} g^{-\frac{4}{5}} = 4 \cdot \frac{4}{\cancel{25}\,5} l^{-\frac{1}{5}} g^{\frac{1}{5}}$

$\Rightarrow$

$$l^{4/5} g^{-4/5} = \frac{16}{5} l^{-\frac{1}{5}} g^{\frac{1}{5}}$$

Multiplying by $g^{4/5} l^{\frac{1}{5}}$, we get,

$$l^{4/5} l^{1/5} \cancel{g^{-4/5}} \cancel{g^{4/5}} = \frac{16}{5} \cancel{l^{-1/5}} \cancel{l^{1/5}} g^{\frac{1}{5}} g^{\frac{4}{5}}$$

$$\Rightarrow \quad l^{\frac{4}{5}+\frac{1}{5}} = \frac{16}{5} g^{\frac{1}{5}+4/5}$$

$$\Rightarrow \quad 5/16\ l = g. \quad --- \quad (4).$$

Plugging into (3), ~~we~~ we get.

$$5l + 4\left(\frac{5}{16}l\right) - 20 = 0$$

$$\Rightarrow \quad 5l + \cancel{4}\frac{5}{4} l = 20$$

$$\Rightarrow \quad \frac{{}^{5}\cancel{25}}{4} l = \cancel{20}\ 4$$

$$\Rightarrow \quad l = \frac{16}{5}$$

Plugging into (4) gives us

$$g = 1.$$

Point $\left(\frac{16}{5}, 1\right)$ Value $= f\left(\frac{16}{5}, 1\right) = \left(\frac{16}{5}\right)^{4/5}.$

Maximum or minimum?

Plug another point, $(l=0, g=5.)$

satisfies $5l+4g=20$

$f(0,5)=0.$

Comparing with what we got, we conclude it is the maximum.

Diagram