25th February (1)

Partial Fractions and Trignometric Substitutions

$$\int_{3/2}^{\frac{3\sqrt{3}}{2}} \frac{x^2}{\sqrt{9-x^2}}\,dx.$$

$9 = 3^2$. The form is.

$\sqrt{9-x^2} = \sqrt{3^2-x^2}$

So $-3 \le x \le 3$.

Let $x = 3\sin\theta$

$dx = \frac{d}{d\theta}(3\sin\theta)\,d\theta$

$= 3\cos\theta\,d\theta$

$a^2 - x^2 \longleftrightarrow x = a\sin\theta;\ |x| \le a$

$a^2 + x^2 \longleftrightarrow x = a\tan\theta$

$x^2 - a^2 \longleftrightarrow x = a\sec\theta;\ |x| \ge a$

- Complete the square.
- Identify the form and make the substitution.
- Use trig. integrals. (or other available techniques)

$x = \frac{3}{2} \Rightarrow 3\sin\theta = \frac{3}{2} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$

$x = \frac{3\sqrt{3}}{2} \Rightarrow 3\sin\theta = \frac{3\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}.$

Substituting we get,

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\sin^2\theta\ 3\cos\theta}{\sqrt{9(1-\sin^2\theta)}}\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\sin^2\theta\ 3\cos\theta}{\sqrt{9\cos^2\theta}}\,d\theta.$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{9\sin^2\theta}{3\cos\theta}\, 3\cos\theta \, d\theta = 9\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta \, d\theta$$

$$= \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\sin^2\theta \, d\theta$$

$$= \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1-\cos 2\theta)\, d\theta$$

$$\left[\begin{array}{c} \int \sin^n\theta \cos^m\theta \, d\theta \\ n=2 \quad m=0 \\ n, m, \text{even} \end{array}\right]$$

$$= \frac{9}{2}\,\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos 2\theta \, d\theta$$

$$= \frac{9}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) - \frac{9}{2}\,\frac{\sin 2\theta}{2}\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{3\pi}{4} - \frac{9}{4}\left(\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right)$$

$$= \frac{3\pi}{4} - \frac{9}{4}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\pi}{4}}$$

Partial Fractions

$$\int \frac{x^4+1}{x^3-2x^2+x}\,dx.$$ Step 1: Long division

$$\cancel{x^3-2x^2+x \,\big)\, x^4 + 1 \,\big(\, x} \qquad \cancel{x^4-2x^3+x^2} \qquad \cancel{2x^3-x^2+}$$

$$
\begin{array}{r l}
x^3-2x^2+x\,\big) & x^4 \qquad\qquad\qquad +1 \;\big(\; x+2 \\
- & x^4-2x^3+x^2 \\
\hline
& \qquad 2x^3-x^2 \qquad +1 \\
- & \qquad 2x^3-4x^2+2x \\
\hline
& \qquad\qquad 3x^2-2x+1.
\end{array}
$$

Then $(x^4+1) = (x^3-2x^2+x)(x+2) + (3x^2-2x+1)$

$$\therefore \int \frac{x^4+1}{x^3-2x^2+x}\,dx = \int \frac{(x^3-2x^2+x)(x+2)+(3x^2-2x+1)}{x^3-2x^2+x}\,dx$$

$$= \int (x+2)\,dx + \int \frac{3x^2-2x+1}{x^3-2x^2+x}\,dx.$$

$$= \frac{x^2}{2}+2x + C + \boxed{\int \frac{3x^2-2x+1}{x^3-2x^2+x}\,dx.}$$

$$\parallel$$
$$?$$

Step 2: Factorise denominator:

$$x^3-2x^2+x = x(x^2-2x+1) = x(x-1)^2.$$

Step 3: Express in form $\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}.$

$$\frac{3x^2-2x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Cross Multiply by $x(x-1)^2$ to get,

$$3x^2-2x+1 = A(x-\boxed{1})^2 + Bx(x-1) + C(x-0)$$

How to find A, B and C? Plug these values for x.

$x=0$ gives ~~3.0 1 = A~~ $3.0-2.0+1 = A(-1)^2$

$\Rightarrow \boxed{1 = A}$

$x=1$ gives $3.1-2.1+1 = \cancel{A}\, C$

$\Rightarrow \boxed{C = 2}$

$x=2$ gives $3.2^2-2.2+1 = A(2-1)^2 + B2(2-1) + C(2-0)$

$\Rightarrow 9 = A + 2B + 2C$

$= 1 + 2B + 4$

$\Rightarrow \boxed{B = 2}$

$$\therefore \frac{3x^2-2x+1}{x(x-1)^2} = \frac{1}{x} + \frac{2}{x-1} + \frac{2}{(x-1)^2}$$

$$\therefore \int \frac{3x^2-2x+1}{x(x-1)^2}\,dx = \int \frac{1}{x}\,dx + \int \frac{2}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx$$

$$= \ln|x| + 2\ln|x-1| + 2\int \frac{dx}{(x-1)^2}$$

$$= \ln|x| + 2\ln|x-1| + 2\left\{\frac{(x-1)^{-1}}{-1}\right\}$$

$$= \ln|x| + 2\ln|x-1| - \frac{2}{x-1}.$$

Thus $\int \frac{x^4+1}{x^3-2x^2+x}\,dx = \frac{x^2}{2} + 2x + \ln|x| + 2\ln|x-1| - \frac{2}{x-1} + C.$

In general, $\frac{p(x)}{(x-r_0)(x-r_1)^2(x-r_2)^3}$ ← polynomial

r_0, r_1, r_2 → numbers like 1, 2, 5 ...

you can write it as

$$\frac{A}{(x-r_0)} + \frac{B}{(x-r_1)} + \frac{C}{(x-r_1)^2} + \frac{D}{(x-r_2)} + \frac{F}{(x-r_2)^2} + \frac{G}{(x-r_2)^3}.$$

$\int \frac{(x-2)}{(x-1)^3}\,dx =$ { No need for long division

$$\frac{x-2}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}.$$

$\Rightarrow x-2 = A(x-1)^2 + B(x-1) + C$ [Multiply by $(x-1)^3$]

Put $x=1 \Rightarrow -1 = A(1-1)^2 + B(1-1) + C$

$\Rightarrow \boxed{C=-1}$

Put $x=0 \Rightarrow 0-2 = A(0-1)^2 + B(0-1) + (-1)$

$\Rightarrow A - B = -1$ — — (1).

Put $x=2 \Rightarrow 2-2 = A(2-1)^2 + B(2-1) + (-1)$

Integration

- Riemann Sums. [$\int x\,dx$, $\int x^2\,dx$, $\int x^3\,dx$] + reverse procedure.
- Area under curves. (simple functions. $\int_1^2 x\,dx$, $\int_0^1 \sqrt{1-x^2}\,dx$)
- Anti derivatives (Fundamental Theorem of Calculus)

  Take derivative of $G(x) = \int_{10}^{x^2} e^{t^2}\,dt$.

- Substitution + Trig. substitution

• Find something nice to replace a part of a function by something else.

- $\int \frac{x^3+1}{x^4+4x}\,dx$, $\int \frac{e^x}{e^x+2}\,dx$, $\int \frac{dx}{\sqrt{x^2-10}}$,

$$\int \frac{dx}{\sqrt{x^2-2x+3}}.$$

- Integration by parts.
  - express as product $\int u\,dv = vu - \int v\,du$.

Follow
L og
I nverse.
A lgebraic
T rignometric
E xponential
D

$\Rightarrow$ $A + B = 1$ --- 2

Adding 1 and 2, we get

$\boxed{A = 0}$

Putting $A = 0$ in 2, we get,

$\boxed{B = 1}$

$$\int \frac{(x-2)}{(x-1)^3}\,dx = \int \frac{dx}{(x-1)^2} - \int \frac{dx}{(x-1)^3}$$

$$= \int (x-1)^{-2}\,dx - \int (x-1)^{-3}\,dx$$

$$= \frac{(x-1)^{-1}}{-2+1} - \frac{(x-1)^{-2}}{-3+1} + C$$

$$= \frac{(x-1)^{-2}}{2} - \frac{(x-1)^{-1}}{1} + C$$

$$= \frac{1}{2(x-1)^2} - \frac{1}{(x-1)} + C.$$

~~xa~~.

— $\int x^2 \sin x \, dx$, $\cancel{\int \tan x \, dx}$, $\int \frac{\ln(x)}{x^{10}} dx$,

$x^{-10} \to$ not inverse function

$\int \arctan x \, dx$, $\int_{2/\sqrt{2}}^{2} z \operatorname{arcsec}(z) \, dz$.

$\cancel{\int_{2/\sqrt{3}}^{2} z \sec}$

— Trig Integrals

~~+ Powers $\int \sin^m x \cos$~~

— $\int \sin^m x \cos^n(x) \, dx$.

— $\int \tan^m(x) \sec^n(x) \, dx$

— $\int \cot^m(x) \operatorname{cosec}^n(x) \, dx$.

— $\int_{\pi/6}^{\pi/3} \cot^3(\theta) \, d\theta$, $\int \sin^{-3/2}(x) \cos^3(x) \, dx$,

$\int_0^{\sqrt{\pi/2}} x \sin^3(x^2) \, dx$.

— Partial Fractions

$\int \frac{p(x)}{q(x)}$ $p(x) \to$ polynomial, $q(x) \to$ polynomial.

$\int \frac{5x \, dx}{(x^2-1)(x-2)}$, $\int \frac{(2x+3) \, dx}{(x-1)^2 (x-5)}$, $\cancel{\int \frac{3x}{(x-1)(x}}$

$\int \frac{3x^2}{x^2-3x+2} dx$.