~~Limits and S~~
Sequences and Series

25th March

A sequence is a list of numbers

$$\{ a_1 \; a_2 \; \ldots \; a_n \; \ldots \} = \text{List}.$$

Examples:

- $a_n = \frac{1}{n^2}$ — formulae.

$$1, \frac{1}{4}, \frac{1}{9} \ldots \qquad 1 = a_1,\ \frac{1}{4} = a_2,\ \frac{1}{9} = a_3$$

- $a_n = (-1)^n$

$$1, -1, 1, -1, \ldots \qquad 1 = a_1,\ -1 = a_2,\ 1 = a_3,\ -1 = a_4$$

- $a_n = \frac{n}{n+1}$

$$\left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots \right. \qquad \frac{1}{2} = a_1,\ \frac{2}{3} = a_2,\ \frac{3}{4} = a_3$$

Recursion. ~~Recurrence~~. a_{i+1} = Some expression in lower a_i's

Example:

- $a_{i+1} = 2a_i;\ a_1 = 2 \Rightarrow a_1 = 2,\ a_2 = 2a_1 = 4,\ a_3 = 2a_2 = 2.4 = 8.$
- $a_{i+1} = (i+1)\,a_i;\ a_1 = 1 \Rightarrow a_1 = 1,\ a_2 = 2.a_1 = 2.1,\ a_3 = 3a_2 = 3.2.1.$
- ~~$a_{i+1} = \frac{a_i}{a_{i-1}} + i;\ a_1 = 4 \Rightarrow a_1 = 4$~~ ~~$a_2 =$~~

$$a_i = i.(i-1) \ldots 1 = i!$$

Three ways a sequence is given —

1) List
2) Formulae
3) Recursion.

Limits

A sequence (a_n) converges to a limit L, if it gets "closer" to L as n grows large.

If limit exists then (a_n) is convergent
If limit does not exist then (a_n) is divergent.

Examples:

Find the limit of the sequence $\{1, \frac{1}{2}, \frac{1}{3} \cdots\}$

Solution: First write general term: a_n.

$a_1 = 1$
$a_2 = \frac{1}{2}$
$a_3 = \frac{1}{3}$ ... guess $a_n = \frac{1}{n}$.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0$$

Find the limit of the sequence $\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \cdots\right\}$.

Solution:

$a_1 = \frac{1}{2} = \frac{1}{1^2+1}$
$a_2 = \frac{2}{5} = \frac{2}{2^2+1}$
$a_3 = \frac{3}{10} = \frac{3}{3^2+1}$
$a_4 = \frac{4}{17} = \frac{4}{4^2+1}$

Guess: $a_n = \frac{n}{n^2+1}$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n}{n^2+1} = \lim_{n \to \infty} \frac{1}{n + \frac{1}{n}}$$ (Divide.

$$= \lim_{n\to\infty} \frac{\frac{n}{n^2}}{\frac{n^2+1}{n^2}} \quad \left[n^2 \text{ is highest power of } n \right]$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}\nearrow 0}{1+\frac{1}{n^2}\nearrow 0} = \frac{0}{0+1} = 0 \quad \left(\begin{matrix} \text{Rule 1} \\ + \text{Rule 3} \\ + \text{Rule 4} \end{matrix} \right)$$

Properties: 1) $\lim_{n\to\infty}(a_n+b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n)$ if they both converge.

2) $\lim_{n\to\infty} c\, a_n = c(\lim_{n\to\infty} a_n)$

3) $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$

4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$. What rules did we use in last problem?

- $\lim_{n\to\infty}$ Find limit of sequence $\{-1, 1, -1, 1, -1, 1 \cdots\}$ if it exists.

Solution: The sequences does not head to any particular number. $(-1, \text{ or } 1)$.

$\therefore$ it diverges.

- Find the limit of the sequence $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots\}$

Solution: ~~genera~~

$a_1 = 1$, $a_2 = -\frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = -\frac{1}{4}$.

Formula: $a_n = (-1)^{n+1}\frac{1}{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{n} = 0$$ [Even thought $\lim_{n\to\infty} (-1)^{n+1}$ does not exist].

Series $\sum_{k=1}^{\infty} a_k$.

Difference from sequences — (summation sign $\sum$)

But they are essentially the same.

Given a series we get a sequence of partial sum

$S_1 = a_1$
$S_2 = a_1 + a_2$
$\vdots$
$S_n = a_1 + a_2 + \dots + a_n$

Such that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\{S_i\}$ converges.

- One way to check convergence ~~and~~ or divergence — integral test.

If we are summing infinitely many things they must be be coming small as well.

Divergence test (How can sequences help in series?)

~~If~~

Theorem: ~~If $\lim_{k\to\infty} a_k$~~

Divergence test: If $\lim_{k\to\infty} a_k = 0$ Test inconclusive.

If $\lim_{k\to\infty} a_k \neq 0 \Rightarrow \sum a_k$ diverges.

or does not exist.

Qn. ~~Do~~ Apply divergence test to.

1) $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$. $\sum_{k=1}^{\infty} (-1)^k k$, $\sum_{k=1}^{\infty} 2^{-k}$, $\sum_{k=1}^{\infty} k \sin\left(\frac{1}{k}\right)$,

Solution: 2) $\sum_{k=1}^{\infty} (-1)^k k$. $a_n = (-1)^n n$

$\lim_{n\to\infty} (-1)^n n$ is not define.

Divergence test $\Rightarrow \sum_{k=1}^{\infty} (-1)^k k$ diverges.

3) $\sum_{k=1}^{\infty} 2^{-k}$. $a_k = 2^{-k}$.

$\lim_{n\to\infty} 2^{-n} = 0$.

~~$\lim_{n\to\infty}$~~ Divergence test non conclusive.

~~Side Notes~~

~~Suppose $\lim_{n\to\infty} a_n = 5$.~~

~~What is $\lim_{n\to\infty} a_n^2 + 5$?~~

~~$= 5^2 + 5 = 30$.~~

~~Back to~~ But how do we judge whether a sequence converges.

Facts one may use: Rules noted on page 3

Use these to break into parts you know how to handle.

1) $\lim_{n\to\infty} \frac{n^2}{1+n^2}$.

Note $\lim_{n\to\infty} n^2 = \infty$

$\lim_{n\to\infty} 1+n^2 = \infty$

$\rightsquigarrow$ Cannot conclude anything.

But $\frac{n^2}{1+n^2}$

$= \frac{1}{\frac{1}{n^2}+1}$.

$n \longrightarrow \infty \Rightarrow n^2 \longrightarrow \infty$

$\Rightarrow \frac{1}{n^2} \longrightarrow 0$.

$\Rightarrow 1 + \frac{1}{n^2} \longrightarrow 1+0=1$

$\Rightarrow \frac{1}{1+\frac{1}{n^2}} \longrightarrow \frac{1}{1+0} = \boxed{1}$

$\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$ diverges.

4) $\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right)$

$$= \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

$$n \longrightarrow \infty \Rightarrow \frac{1}{n} \longrightarrow 0$$

$$\Rightarrow \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \longrightarrow \boxed{1} \quad \left[\lim_{x\to 0} \frac{\sin(x)}{x} = 1\right]$$

~~Squeeze Theorem~~

$$\cancel{\left(\sum \frac{\sin(n)}{n}\right)}$$

Thus $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ diverges.

Squeeze Theorem