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Differential Equations (Application of integrals - I) 27th February

$y = 10x - 5y$ → We can find y in times of x.

Equation in y and x.

$\Rightarrow 6y = 10x$

$\Rightarrow y = \frac{5x}{3}$

$y' = 10x - 5y$

→ Equation with derivatives → differential Equations (DE).

How do we solve them?

We will solve some simple ~~diffe~~ differential. equations called ~~to~~ separable differential equations (DE).

How to solve them?

$y' = (12 - 5y)$, $y' \rightarrow \frac{dy}{dt}$

[Assume $12 - 5y \neq 0$]

$\frac{dy}{dt} = (12 - 5y)$

$\Rightarrow \int \frac{dy}{12-5y} = \int 1\,dt$

$= t + c_1$

$\Rightarrow \int \frac{dy}{12-5y}$

Substituting

$u = 12 - 5y$

$du = -5\,dy$

$$= \int \frac{\left(\frac{du}{-5}\right)}{u} = -\frac{1}{5}\int \frac{du}{u} = -\frac{1}{5}\ln|u| + C_2$$

$$= -\frac{1}{5}\ln|12-5y| + C_2$$

Thus, $-\frac{1}{5}\ln|12-5y| + C_2 = t + C_1$

$$\Rightarrow -\frac{1}{5}\ln|12-5y| = t + (C_1 - C_2) = C_4$$

$$\Rightarrow \cancel{-\frac{1}{5}} \ln|12-5y| = -5\{t + C_4\}$$

$$\Rightarrow e^{\ln|12-5y|} = e^{-5t-5C_4}$$

$$\Rightarrow |12-5y| = e^{-5t}\underbrace{e^{-5C_4}}_{C_3}$$

Let $12-5y>0$, then

$$12-5y = C_3 e^{-5t}$$

$$\Rightarrow y = \frac{12 - C_3 e^{-5t}}{5} = \frac{12}{5} - \underbrace{\left(\frac{C_3}{5}\right)}_{=C} e^{-5t}$$

$$= \underline{\frac{12}{5} - C e^{-5t}}$$

→ Solution.

Note $C>0$

Similarly $12-5y<0$ will yield $C<0$; $12-5y=0$, will give $C=0$.

How to determine C?

Initial value problem ↱ given one value

Solve for $y' = (12-5y)$ $\boxed{y(1) = 1}$

Solution from last problem

$$y = \frac{12}{5} - Ce^{-5t}$$

$$y(1) = \frac{12}{5} - Ce^{-5} = 1$$

$$\Rightarrow \quad Ce^{-5} = \frac{7}{5}$$

$$\Rightarrow \quad C = \frac{7}{5}e^{5}$$

Then solution is. $y = \frac{12}{5} - \frac{7}{5}e^{5}e^{-5t}$

- There were no ants in Tanzania at one point. ~~At one point~~ 10 ants were brought to the country by a ~~shif~~ ship. ~~Take y as the number of ants, ie ant population.~~

Ant population y satisfies equation.

$y' = 2ty + y$. where t is time in days ~~minutes~~

Find ant population after 20 days.

Solution: $y(0) = 10$.

$$y' = 2ty + y$$
$$\overset{\parallel}{\frac{dy}{dt}}$$

$$\Rightarrow \boxed{\frac{dy}{dt} = y(2t+1)} \rightarrow \text{separate } y \text{ and } t.$$

$$\Rightarrow \int \frac{dy}{y} = \int (2t+1)\,dt$$

$$\Rightarrow \ln|y| = t^2 + t + c_1.$$

$$\Rightarrow \cancel{|y| =} \; e^{\ln|y|} = e^{t^2+t+c_1}$$

$$\Rightarrow |y| = e^{t^2+t+c_1}$$
$$= \underbrace{(e^{c_1})}_{C} e^{t^2+t}$$

Number of ants $= y > 0$. Thus $|y| = y$.

$$\Rightarrow y(t) = C e^{t^2+t} \quad [\text{Replace } y \text{ by } y(t)]$$

$$y(0) = 10 \Rightarrow C e^{(0)^2+(0)} = 10$$
$$\Rightarrow C e^{0} = 10$$
$$\Rightarrow C = 10. \quad [e^0 = 1]$$

Thus. $y(t) = 10\, e^{t^2+t}$.

After 20 days.

$y(20) = 10\, e^{(20)^2 + 70} = 10\, e^{470}$.

$$\int_{-1}^{1} \frac{1}{y^2}\, dy = \int_{-1}^{1} y^{-2}\, dy = \frac{y^{-2+1}}{-2+1}\Bigg|_{-1}^{1}$$

$$\cancel{= \left\{ y^{-1}\Big|_{-1}^{1} \right\}}$$

$$= \frac{1^{-1}}{-1} - \left(\frac{(-1)^{-1}}{-1}\right)$$

$$= -1 - (1)$$

$$= -2.$$

~~But~~ But $y^2 > 0 \Rightarrow \frac{1}{y^2} > 0$ everywhere.

Should not $\int \frac{1}{y^2} dy > 0$?

$\frac{1}{y^2}$ is not defined at $y = 0$.

Every thing we learnt was for continuous functions. $\frac{1}{y^2} \to \infty$ as $y \to 0$.

Idea. $\int_{-1}^{1} \frac{1}{y^2}\, dy \cancel{= \lim_{c \to 0} \int_{-1}^{-c} \frac{1}{y^2}\, dy + \lim_{c \to 0} \int_{c}^{1} \frac{1}{y^2}\, dy}$.

$$\cancel{= \lim_{c \to 0^-} \int_{-1}^{-c} \frac{1}{y^2}\, dy} = \lim_{c \to 0^-} \int_{-1}^{c} \frac{1}{y^2}\, dy + \lim_{c \to 0^+} \int_{c}^{1} \frac{1}{y^2}\, dy.$$

Improper Integrals with Unbounded Integrand

- f is continuous on $(a,b]$ with

$$\lim_{x\to a^+} f(x) = \pm\infty$$

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$ if the limit exists.

- f is continuous on $[a,b)$ with

$$\cancel{\lim_{x\to b^-}} \quad \lim_{x\to b^-} f(x) = \pm\infty$$

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$ if the limit exists.

- f is continuous on $[a,b]$ except at a point p [0 in the last problem]

$$\lim_{x\to p^\pm} f(x) = \pm\infty$$

$$\int_a^b f(x)\,dx = \lim_{c\to p^-} \int_a^c f(x)\,dx + \lim_{c\to p^+} \int_c^b f(x)\,dx.$$

$$\int_{-1}^{1} \frac{1}{y^2}\,dy = \cancel{\lim} \quad \cancel{\text{discontinuity}}.$$

$$\lim_{y \to 0^{\pm}} \frac{1}{y^2} = \infty$$

$\frac{1}{y^2}$ is continuous elsewhere.

Thus

$$\int_{-1}^{1} \frac{1}{y^2}\,dy = \lim_{c \to 0^-} \int_{-1}^{c} \frac{1}{y^2}\,dy + \lim_{c \to 0^+} \int_{c}^{1} \frac{1}{y^2}\,dy$$

$$\cancel{= \lim_{c \to 0^-} -\frac{1}{y}}$$

$$= \lim_{c \to 0^-} \left\{ -\frac{1}{y}\Big|_{-1}^{c} \right\} + \lim_{c \to 0^+} \left\{ -\frac{1}{y}\Big|_{c}^{1} \right\}.$$

$$= \lim_{c \to 0^-} \left\{ -\frac{1}{c} - \left(\frac{1}{-1}\right) \right\} + \lim_{c \to 0^+} \left\{ -\frac{1}{1} + \left(\frac{1}{c}\right) \right\}$$

$$\lim_{c \to 0^-} -\frac{1}{c} = -\lim_{c \to 0^-} \left(\frac{1}{c}\right) = -(-\infty) = \infty.$$

$$\lim_{c \to 0^+} \frac{1}{c} = \infty.$$

$$\therefore \int_{-1}^{1} \frac{1}{y^2}\,dy = \infty + \infty = \infty \quad \left[\text{Note } \infty - \infty \text{ is undefined}\right]$$

~~$\lim \int_0^1 \frac{1}{\sqrt{x}}\,dx$~~ $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty$. It is continuous elsewhere on $(0,1)$.

$$\therefore \int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c \to 0^+} \int_c^1 \frac{1}{x^{\frac{1}{2}}}\,dx$$

$$= \lim_{c \to 0^+} \left. \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right|_c^1$$

$$= \lim_{c \to 0^+} \left\{ \frac{1^{-\frac{1}{2}+1}}{\frac{1}{2}} - \frac{c^{-\frac{1}{2}+1}}{\frac{1}{2}} \right\}$$

$$= \lim_{c \to 0^+} \left\{ 2 - 2c^{\frac{1}{2}} \right\}$$

$\lim_{c \to 0}$

$$= 2 - 2\lim_{c \to 0^+} c^{\frac{1}{2}} = 2.$$