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Squeeze Theorem and Series — 27th March

~~Read~~

Recall: Suppose $\sum_{n=1}^{\infty} a_n$ is a series.

~~Then~~ If ~~tant~~ $\lim_{n\to\infty} a_n \neq 0$ then

$\sum_{n=1}^{\infty} a_n$ diverges. — Divergence test.

Note: $\cancel{\sum_{n=1}^{\infty} a_n}$ $\lim_{n\to\infty} a_n = 0$ is inconclusive.

- $\sum_{n=1}^{\infty} \frac{\sin n}{n}$

Divergence test: $\lim_{n\to\infty} \frac{\sin n}{n}$.

Now $-1 \leq \sin(n) \leq 1$

Thus $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

~~Thus~~ But $\lim_{n\to\infty} -\left(\frac{1}{n}\right) = 0$

$\lim_{n\to\infty} \frac{1}{n} = 0$.

$\therefore \lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

— Squeeze Theorem.

Thus divergence test is inconclusive

- If a_n, b_n, c_n are sequences and $a_n \leq b_n \leq c_n$. Then $\lim_{n\to\infty} a_n = L$

$\lim_{n\to\infty} c_n = L$

$$\boxed{\Rightarrow \lim_{n\to\infty} b_n = L}$$

In the previous example.

$a_n = -\frac{1}{n}$, $c_n = \frac{1}{n}$, $b_n = \frac{\sin n}{n}$, $L = 0$.

Geometric series.

We say Divergence test was inconclusive

for ~~$a_n = 2^{-n}$~~. $\sum_{n=1}^{\infty} 2^{-n}$

Geometric Series $\sum_{n=1}^{\infty} a r^{n-1}$.

$\boxed{\sum_{n=1}^{\infty} 2\, 3^{n-1},\ \sum_{n=1}^{\infty} 2\left(\frac{3}{4}\right)^{n-1},\ \sum_{n=1}^{\infty} 2^n 3^{-n}}$

a: First term $= a_0$

r: Common ratio $= \frac{a_1}{a_0}$.

Let us apply divergence test to this.

$$\lim_{n\to\infty} a r^{n-1} = a \lim_{n\to\infty} r^{n-1}$$

If $-1 < r < 1$, Then $\lim_{n\to\infty} r^{n-1} = 0$.

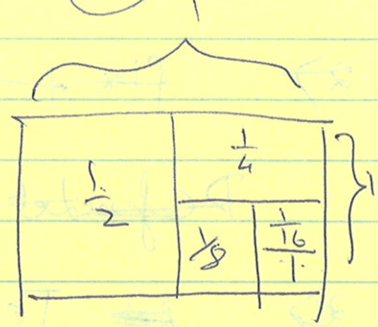
test inconclusive.

If $r \geq 1$ or $r \leq -1$ the $\lim_{n\to\infty} r^{n-1} \neq 0$

$\Rightarrow \sum a_n r^{n-1}$ is divergent.

Thus we care about ~~sequence~~ series

$$\sum_{n=1}^{\infty} a r^{n-1} \text{ where } -1 < r < 1.$$



$\cancel{\sum_{n=x}}$ We checked by integral test

$$\sum_{n=1}^{\infty} 2^{-n} \text{ converges.}$$

But what does it add up to?

$$\boxed{\sum_{n=1}^{\infty} a r^{n-1}} = a \sum_{n=1}^{\infty} r^{n-1}$$

$$= a \cdot \lim_{n \to \infty} \left(\sum_{k=1}^{n} r^{k-1} \right)$$

$$= a \lim_{n \to \infty} \left(\frac{r^n - 1}{r - 1} \right) \longrightarrow \underline{\text{Formula}}$$

$$= a \frac{\lim_{n \to \infty} (r^n) - 1}{r - 1}$$

$$= \frac{a(-1)}{(r-1)} = \boxed{\frac{a}{1-r}}$$

[a is the first term, r is the common ratio]

$\underline{\text{Qn}}$ $$\sum_{n=1}^{\infty} (10)^{1-n} 9^n = 10 \sum_{n=1}^{\infty} 10^{-n} 9^n$$

$$= 10 \sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^{n}$$

$$= 10 \sum_{n=1}^{\infty} \left(\frac{9}{10}\right) \left(\frac{9}{10}\right)^{n-1} \quad \left[a = \frac{9}{10} \cdot r = \frac{9}{10} \right] \quad \left(\frac{9}{10}\right) < 1$$

$$\cancel{= 10 \cdot \frac{9}{10} \sum_{n=1}^{\infty}}$$

$$= \cancel{\left(10 \cdot \frac{9}{10}\right)} 10 \left(\frac{9}{10}\right) \left(\frac{1}{1 - \frac{9}{10}}\right) = 10 \left(\frac{9}{10} \left(\frac{1}{\frac{1}{10}}\right)\right)$$

$$= 90$$

27) ~~f(x f: (-1, 1)~~

~~Define Let $f(x) = \sum_{x=0}^{\infty} x^{2n}$~~.

28) Telescopic Series.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+3)}$$ ⇝ do partial fraction.

$$\frac{1}{k(k+3)} = \frac{A}{k} + \frac{B}{k+3}.$$

Then ~~exercise~~: $A = \frac{1}{3}$, $B = -\frac{1}{3}$

[Exercise].

$$\sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \cancel{\sum_{k=1}^{\infty} \frac{1}{k+3} - \frac{1}{k}}.$$

$$= \sum_{k=1}^{\infty} \frac{\frac{1}{3}}{k} + \frac{-\frac{1}{3}}{k+3} = \sum_{k=1}^{\infty} \frac{1}{3}\left(\frac{1}{k} - \frac{1}{k+3}\right)$$

$$= \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+3}\right)$$

Divergence test.

$$\lim_{k \to \infty} \left(\frac{1}{k} - \frac{1}{k+3}\right) = 0$$. Inconclusive Test.

⇝ write it out.

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+3}\right) = \lim_{n\to\infty} \boxed{\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+3}\right)}$$

<u>$n=1$</u>

$$\left(\frac{1}{1}-\frac{1}{1+3}\right) = 1-\frac{1}{4}$$

<u>$n=2$</u>

$$\left(\frac{1}{1}-\frac{1}{1+3}\right) + \left(\frac{1}{2}-\frac{1}{2+3}\right) = 1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}$$

$n=6$

$$\begin{aligned}
&\left(\frac{1}{1}-\cancel{\frac{1}{1+3}}\right)\\
+&\left(\frac{1}{2}-\cancel{\frac{1}{2+3}}\right)\\
+&\left(\frac{1}{3}-\cancel{\frac{1}{3+3}}\right)\\
+&\left(\cancel{\frac{1}{4}}-\frac{1}{4+3}\right) = \left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{7}-\frac{1}{8}-\frac{1}{9}\right]\\
+&\left(\cancel{\frac{1}{5}}-\frac{1}{5+3}\right) \qquad \cancel{-\frac{1}{7}-\frac{1}{8}}\\
+&\left(\cancel{\frac{1}{6}}-\frac{1}{6+3}\right) \qquad \cancel{-\frac{1}{9}\Big)}
\end{aligned}$$

<u>In general</u>

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$$\cancel{\sum_{k=1}^{\infty}} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+3}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{1+3}}\right)$$

$$+ \left(\frac{1}{2} - \cancel{\frac{1}{2+3}}\right)$$

$$+ \left(\frac{1}{3} - \cancel{\frac{1}{3+3}}\right)$$

$$+ \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{4+3}}\right)$$

$$\vdots$$

$$+ \left(\cancel{\frac{1}{n-2}} - \frac{1}{n+1}\right)$$

$$+ \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+2}\right)$$

$$+ \left(\cancel{\frac{1}{n}} - \frac{1}{n+3}\right)$$

$$\cancel{+ \left(\frac{1}{n+1}\right.}$$

$$= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

~~lim~~

$$\sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \frac{1}{3} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+3}\right)$$

$$= \frac{1}{3} \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+3}\right)$$

$$= \frac{1}{3} \lim_{n \to \infty} \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right)$$

$$= \frac{1}{3}\left[1 + \frac{1}{2} + \frac{1}{3}\right]$$

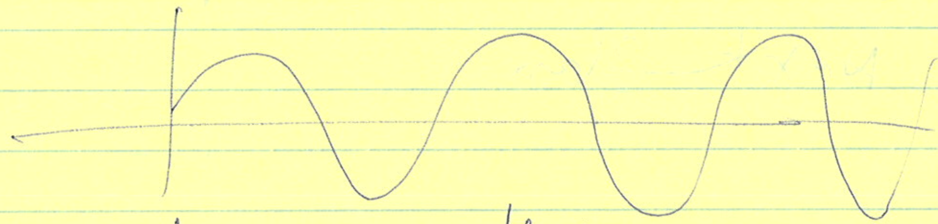
$$= \sin(1) - \lim_{n\to\infty} \sin\left(\frac{1}{n+1}\right) = \sin(1).$$

~~Ratio test~~ & ~~Comparison test~~.

~~Next~~ Class.

- $\sum_{k=1}^{\infty}\left(\sin(n)-(\sin(n+1)\right)$

Divergence test $\lim_{n\to\infty}\left(\sin(n)-\sin(n+1)\right)$



always oscillatin as n tends to infinity

Can be proved $\lim_{n\to\infty}\left(\sin(n)-\sin(n+1)\right)$ does not exist.

~~Series~~ ~~Diverges~~ Thus $\sum_{k=1}^{\infty}\sin(n)-\sin(n+1)$ diverges

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- $\sum_{k=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)$

Divergence test:

$\lim_{n\to\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)=0$ Test is inconclusive.

$$\sum_{k=1}^{\infty}\left(\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right)$$

$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left[\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)\right)$$

$$=\lim_{n\to\infty}\left(\sin\left(\frac{1}{1}\right)-\cancel{\sin\left(\frac{1}{2}\right)}\right)+\left(\cancel{\sin\left(\frac{1}{2}\right)}-\cancel{\sin\left(\frac{1}{3}\right)}\right)+\dots+\left(\cancel{\sin\left(\frac{1}{n}\right)}-\sin\left(\frac{1}{n+1}\right)\right)$$

$$=\lim_{n\to\infty}\left[\sin(1)-\sin\left(\frac{1}{n+1}\right)\right]$$

Monotone sequences

Increasing sequence. $a_m \geq a_n$ if $m > n$.

— $\{a_n\} = \{n\}$.

Decreasing sequence. $a_m \leq a_n$ if ~~$m < n$~~. $m > n$.

— $\{a_n\} = \{\frac{1}{n}\}$

Bounded Sequences There exists L, U such that

$$L < a_n < U$$

$$\{a_n\} = \{\sin(n)\} \quad \begin{bmatrix} L = -1 \\ U = 1 \end{bmatrix}.$$

Theorem: Bounded monotone sequences **converge**.
