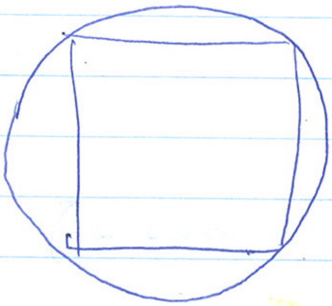
28th January

Integration — Interpretation as an area
— As an antiderivative.
— Techniques.



Idea of Integration

Area of inscribed n-polygon.

4-gon $\approx$ Area is 2.

20 gon ~ 3.09.

100-gon ~ 3.14

Look up digits of pi

Some notation

Σ — sigma.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

end index — n

object to be summed — a_i

index of summation — i

start index — 1

$$\sum_{i=1}^{n} a_i = 1 + 2 + \cdots + n.$$

$$\sum_{i=1}^{n} i^2 = 0^2 + 1^2 + \cdots + 10^2$$

$$\sum_{i=0}^{15} 5 = \underbrace{5 + \cdots + 5}_{16 \text{ times}}.$$

$$\sum_{x=0}^{17} y = \underbrace{y + y + \cdots + y}_{18 \text{ times}} = 18y.$$

Examples:

- $$\sum_{i=1}^{15} 2i = 2\sum_{i=1}^{15} i \quad [\text{By } 1]$$
$$= 2\,\frac{(15)(16)}{2} \quad [\text{By } 4]$$

- $$\sum_{j=1}^{20} (2j + j^3) = 2\sum_{j=1}^{20} j + \sum_{j=1}^{20} j^3 \quad (\text{By } 1 \text{ and } 2]$$
$$= 2\,\frac{(20)(21)}{2} + \frac{(20)^3(20+1)^3}{4} \quad \left(\text{By } 4 \text{ and } 6.\right)$$

- $$\sum_{k=1}^{200} 3\cos(2\pi k) = 3\sum_{k=1}^{200} \cos(2\pi k) = 3\sum_{k=1}^{200} 1$$

~~Reverse:~~ $\cancel{1+3+5+7+9+11 = \sum^{5} (2k+1)}$

$$= 3.200 = 600.$$

2.5

Going from sums to summation notation.

- $\underbrace{1+3+5+7+9+11}\cdots$

Notice they are a sequence of odd numbers, that is of the type $2k+1$.

Starts at $k=0$ and ends at $k=$ ~~10~~ 5

$$1+3+5+\cdots+11=\sum_{k=0}^{5}(2k+1)$$

- $2+4+6+8+\cdots+20$

$$=2(1+2+3+\cdots10)$$

$$=2\left(\sum_{i=1}^{10} i\right)$$

- $1+4+9+16+25+\cdots+100$

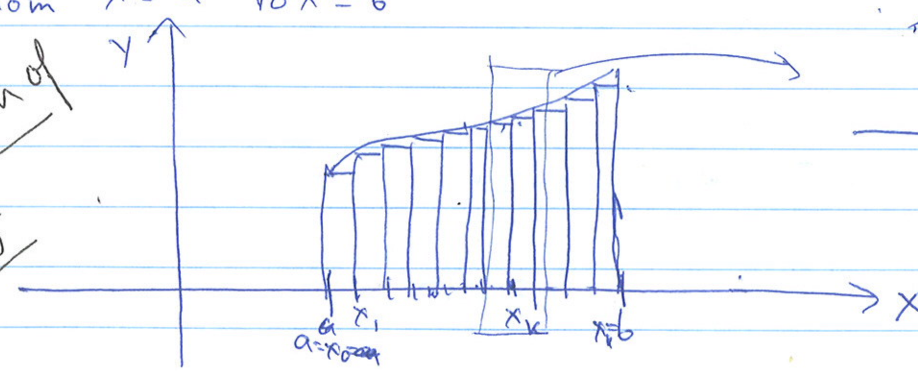
Notice they are a sequence of squares of numbers, that is, of type - k^2

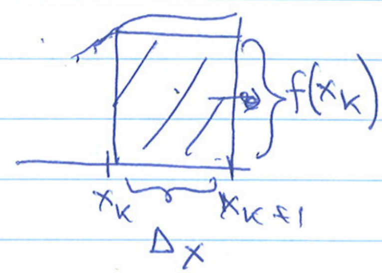
Starts at $k=1$ and ends at $k=10$.

$$1+4+9+16+\cdots100=\sum_{k=1}^{10}k^2.$$

Suppose we want to find area under the curve
$y = f(x)$ from $x = a$ to $x = b$

Graph of f





Divide into n strips ~~of~~ $S_1, S_2, \ldots, S_n$ of equal width $\Delta x = \frac{b-a}{n}$ given by

$[x_0, x_1], [x_1, x_2] \cdots [x_{n-1}, x_n]$.

where $x_0 = a$

$x_1 = a + \Delta x$

$\vdots$

$x_k = a + k\Delta x$

$x_n = b = a + n\Delta x$.

This is called the <u>regular partition</u> of $[a, b]$

Now approximate area by, $S_1 + \cdots S_n = \sum_{i=1}^{n} S_i$

Find out about this person

by the $\boxed{\text{Riemann}}$ sum

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_{n-1})\Delta x$$

$$= \Delta x \sum_{k=1}^{n} f(x_k^*).$$

<u>Left Riemann Sum</u> $x_k^* = x_{k-1}$

$$f(x_0)\Delta_x + f(x_1)\Delta_x + \cdots + f(x_{n-1})\Delta_x$$

$$= \Delta_x \sum_{k=1}^{n} f(x_{k-1}) = \Delta_x \sum_{k=1}^{n} f(a + (k-1)\Delta_x)$$

<u>Right Riemann Sum</u> $x_k^* = x_k$

$$f(x_1)\Delta_x + f(x_2)\Delta_x + \cdots + f(x_n)\Delta_x$$

$$= \Delta_x \sum_{k=1}^{n} f(x_k) = \Delta_x \sum_{k=1}^{n} f(a + k\Delta_x)$$

Midpoint Riemann Sum $\qquad x_k^* = \frac{x_{k-1} + x_k}{2}$

$$\cancel{f(x_0}\; f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)$$

$$\cancel{= \Delta_x \sum_{k=1}^{n} f(x_{k+1}) = \Delta_x \sum_{k=1}^{n} f(}$$

$$= \Delta_x \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) = \Delta_x \sum_{k=1}^{n} f\left(a + \left(\frac{2k-1}{2}\right)\Delta_x\right)$$

Find the left Riemann Sum of $\ln x$ on $[1, 20]$ using $n = 50$.

Solution: $f(x) = \ln(x)$, ~~from~~

$a = 1$

$b = 20$

$\Delta_x = \frac{20-1}{50} = \frac{19}{50}.$

$x_k = a + k\Delta x = 1 + \frac{19k}{50} \qquad k = 0, 1, \ldots 50.$

Left Riemann Sum is

$$\Delta_x \sum_{k=1}^{n} f\left(a + (k-1)\Delta_x\right)$$

$$= \frac{19}{50} \sum_{k=1}^{50} \ln\left(1 + \frac{19}{50}(k-1)\right)$$

- Find the middle Riemann Sum for $f(x) = 1 + x^3$ on $[1, 4]$ using ~~n=4~~ $n=3$

$f(x) = 1 + x^3 \qquad a = 1 \qquad b = 4.$

$\Delta_x = \frac{b-a}{n} = \frac{4-1}{3} = 1$

$x_k = a + k\Delta_x = 1 + k. \qquad k = 0, 1, 2, 3.$

Middle Riemann Sum is.

$\Delta_x \sum_{k=1}^{n} f(\cancel{x_{k-1}} + a + \frac{2k-1}{2}\Delta_x)$

$= 1 \sum_{k=1}^{3} f(1 + \frac{2k-1}{2})$

$= 1 \sum_{k=1}^{3} f(\frac{2k+1}{2})$

$= (1 + (\frac{3}{2})^3) + (1 + (\frac{5}{2})^3) + (1 + (\frac{7}{2})^3).$

Reverse: $\sum_{k=1}^{50} \frac{1}{n\sqrt{\{1 + (\frac{k}{n})^2\}}}$ is the Riemann Sum of which function?

Compare with $\Delta_x \sum_{k=1}^{n} f(x_k^*)$

① Find n and Δ_x

- $n = 50$ $\qquad \cancel{\Delta_x = \frac{b-a}{n}} =$
- Δ_x

Reverse: $\sum_{k=1}^{50} \frac{1}{\boxed{50}\{1+(\frac{k}{50})^2\}}$ is the Riemann sum of which function?

Compare with $\boxed{\Delta_x} \sum_{k=1}^{n} f(x_k^*)$

① Find n and Δ_x $\quad n = 50$

$$\Delta_x = \left(\frac{b-a}{n}\right) = \frac{1}{50}$$

② Recognise function and x_k^* $\quad f(x_k^*) = \frac{1}{1+(\boxed{\frac{k}{n}})^2} \to x_k^*$

$$x_k^* = \frac{k}{n} \quad \text{--- ①}$$

③ Left Riemann Sum: $x_k^* = x_{k-1} = a + (k-1)\Delta_x$
$= a + \frac{(k-1)}{n}$ ✗

Right Riemann Sum. ~~$x_k^* = x_k = a +$~~

$$x_k^* = x_k = a + k\Delta_x$$

$$= a + \frac{k}{n} \quad \text{--- ②}$$ ✓

It is a right Riemann Sum.

④ Find a and b

Comparing $a+\frac{k}{n}$ and $\frac{k}{n}$ in ① and ②,

we get $a = 0$

$$b = x_n = a + \Delta_x n = 0 + \frac{1}{50}50$$
$$= 1$$

⑤ Find the function $f(x_k^*) = f\left(\frac{k}{n}\right) = \frac{1}{1+(\frac{k}{n})^2}$

$$f(x) = \frac{1}{1+x^2}.$$

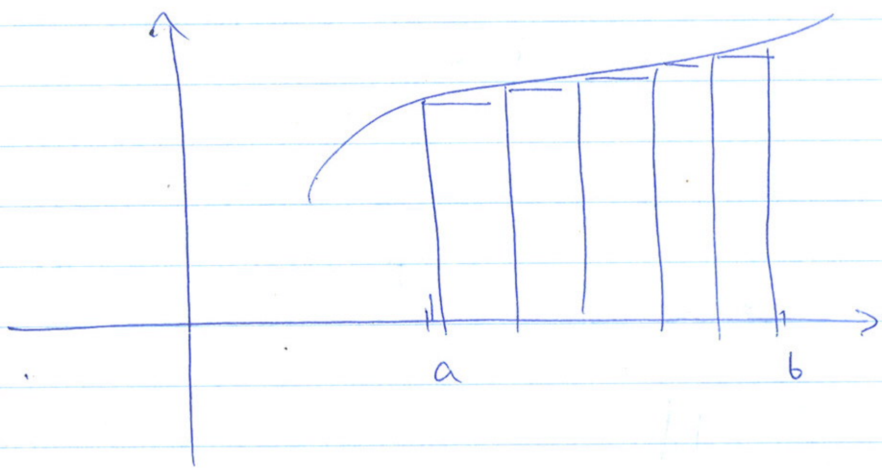
Thus it is the right Riemann Sum for

$f(x) = \frac{1}{1+x^2}$ with ~~n = 0~~ $n = 50$ ~~as the~~

on the interval $[0, 1]$.

But how do we find the area.

$\Delta x \sum_{k=1}^{n} f(x_k^*)$ approximates the area.



How do we find the area?

Take the limit. as $n \to \infty$

The definite integral of the function f from a to b (or on $[a, b]$ is defined as

Area — $\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(x_k^*)$ — Limit of a sum

Integral

Antiderivative.

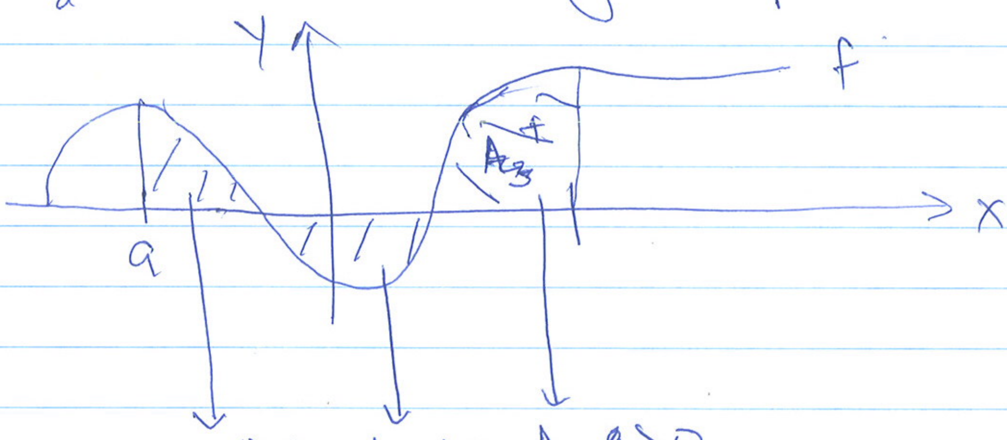
Left Riemann Sum $= \lim_{n\to\infty} \Delta_x \sum_{k=1}^{n} f(x_{k-1})$

Right Riemann Sum $= \lim_{n\to\infty} \Delta_x \sum_{k=1}^{n} f(x_k)$.

So $\int_a^b f(x)\,dx$ represents area under the function from a to b.

What if f is negative?

$\int_a^b f(x)\,dx$. strictly represents signed area



$A_1 > 0 \quad A_2 < 0 \quad A_3 > 0$.

$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3.$$

$\int_a^b |f(x)|\,dx = A_1 + A_2 + A_3$ represents the area.