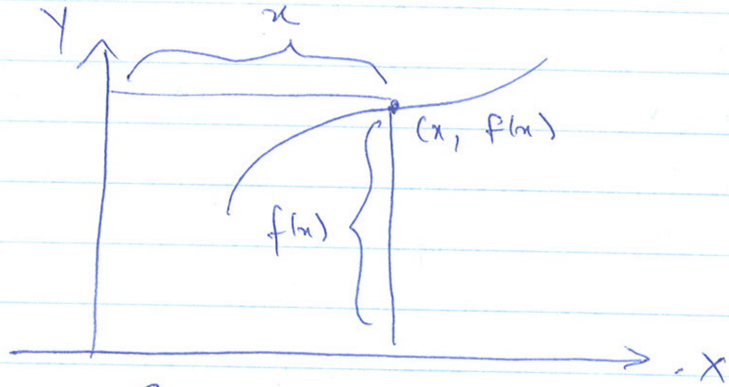
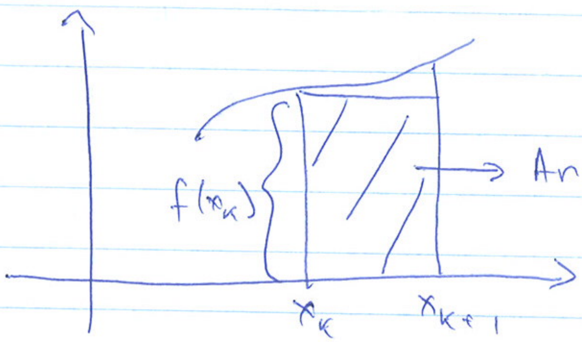
Geometry and Integrals

30th January

Some review

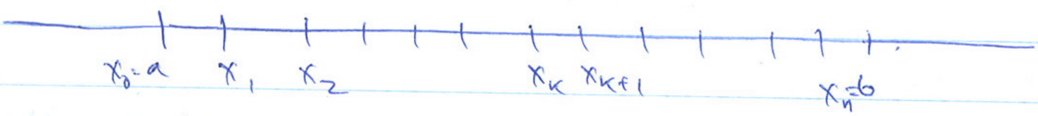


Graph ~~to~~ of f



Want to find area underneath graph of f ~~from a to b~~ on $[a, b]$

Divide $[a,b]$ into n - intervals of width $\Delta_x = \frac{b-a}{n}$
$[x_0\ x_1], [x_1\ x_2]$ - - - - $[x_{n-1}, x_n]$.



where $x_0 = a$

$x_1 = a + \Delta_x$

$x_2 = a + 2\Delta_x$

$\vdots$

$x_k = a + k\Delta_x$

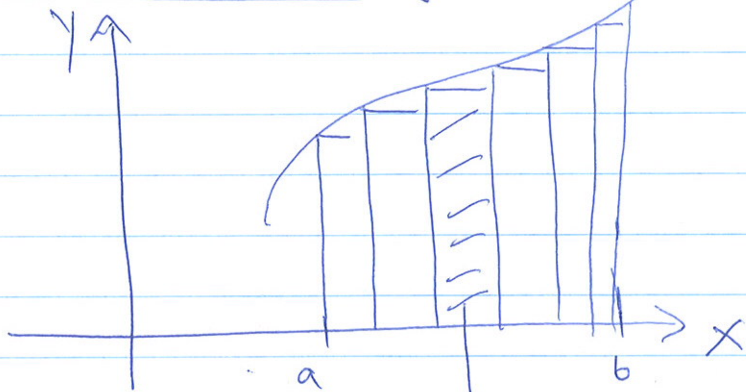
$\vdots$

$x_n = a + n\Delta_x = b$.

Draw rectangles of length $f(x_k^*)$ for interval $[x_{k-1}, x_k]$

~~Left Riem.~~

- Left Riemann Sum (or ~~u~~ using left endpoints)



Area of a rectangle is

$\Delta_x f(x_k^*) = \Delta_x f(x_{k-1})$

Left Riemann sum is $\sum_{k=1}^{n} \Delta_x f(x_{k-1})$

~~or (left endpoints)~~

Similarly for right Riemann sum, we ~~In general~~ will have. $\sum_{k=1}^{n} \Delta_x f(x_k)$

- In general; $\sum_{k=1}^{n} \Delta_x f(x_k^*)$

– Total area under the curve is.

upper limit ← $\boxed{b}$

$$\int_a^{\boxed{b}} \boxed{f(x)}\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta_x f(x_k^*)$$

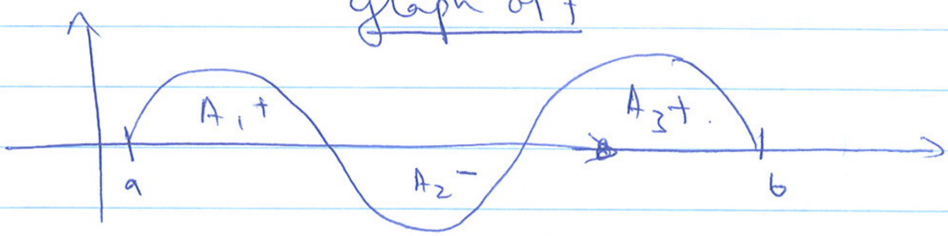
lower limit ← a

integrand.

[Fermat's Idea.]

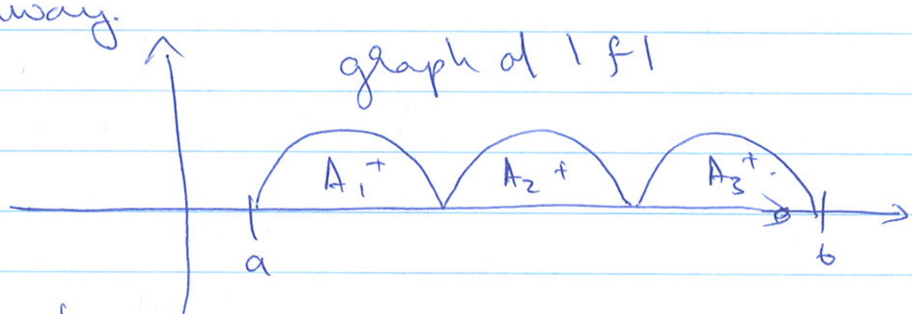
"The integral of f from a to b."

~~If~~ f is negative.



$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 \longleftrightarrow \text{Idea of 'signed' area.}$$

$|f| \geq 0$ always.



$$\int_a^b f(x)\,dx = A_1 + A_2 + A_3.$$

- Suppose $\int_1^2 f(x)\,dx = 5$, $\int_2^5 f(x)\,dx = 7$, $\int_7^5 f(x)\,dx = 3$

What is $\int_1^7 f(x)\,dx$? Look at the Handout.

Using (5), $\int_1^7 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^5 f(x)\,dx + \int_5^7 f(x)\,dx$?

Using (2), $\int_7^5 f(x)\,dx = -\int_5^7 f(x)\,dx = -3.$

~~Thus~~ Thus. $\int_1^7 f(x)\,dx = 5 + 7 + (-3) = 9.$

Qn. Compute the integral.

$$\cancel{\int_0^1 (2\sqrt{1-x^2} + 5)\, dx}.$$

$$\int_0^1 \left(2\sqrt{1-x^2} + 5\right) dx.$$

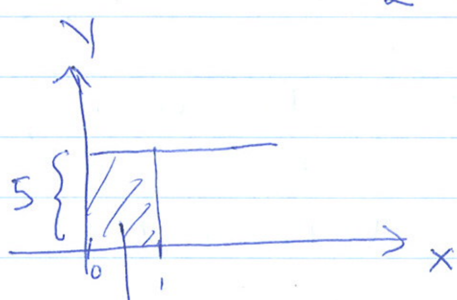
Solution:

By properties 3 and 4, we get

$$\int_0^1 (2\sqrt{1-x^2} + 5)\, dx = 2\underbrace{\int_0^1 \sqrt{1-x^2}\, dx}_{P_1} + \underbrace{\int_0^1 5\, dx}_{P_2}$$

$$= 2P_1 + P_2.$$

$$P_2 = \int_0^1 5\, dx.$$



Area is $5 \times 1 = 5$.

Thus the integral $P_2 = \int_0^1 5\, dx = 5$.

How do we find P_1?

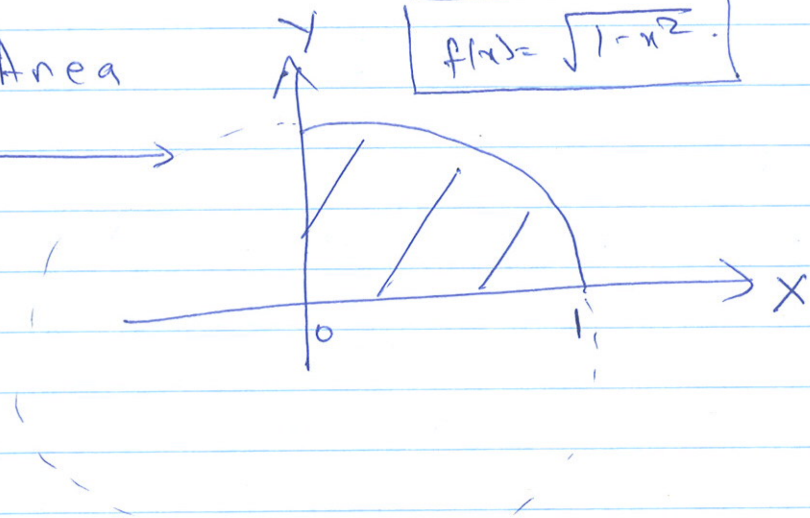
* Use geometry.

$y = \sqrt{1-x^2}$

$\Rightarrow \quad y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1 \quad \boxed{\text{Circle.}}$

$P_1 = \int_0^1 \sqrt{1-x^2}\, dx$ = Shaded Area

$\boxed{f(x) = \sqrt{1-x^2}}$



Area of ~~circle~~ Circle is π.

Area of shaded part $= \frac{\pi}{4} = P_1$

$\therefore \quad \int_0^1 \left(2\sqrt{1-x^2} + 5\right) dx = 2P_1 + P_2 = 2\frac{\pi}{4} + 5.$

- Compute $\int_2^5 (5x + x^2)\, dx$.

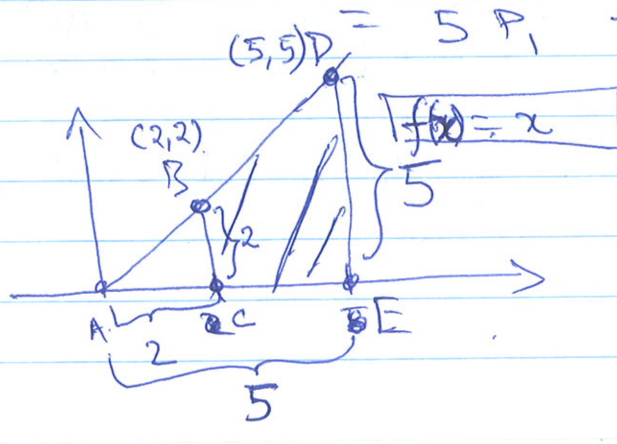
Solution: By 3 $\quad \int_2^5 (5x + x^2)\, dx$

$= \int_2^5 5x\, dx + \int_2^5 x^2\, dx.$

By 4 $\quad = 5\int_2^5 x\, dx + \int_2^5 x^2\, dx.$

$= 5P_1 + P_2$

Find P_1:



$$\int_2^5 x\,dx = \text{Area}(\Delta ADE) - \text{Area}(\Delta ABC)$$

$$= \frac{1}{2} AE \cdot ED - \frac{1}{2} AC \cdot CB.$$

$$= \frac{1}{2}\, 5 \cdot 5 - \frac{1}{2}\, 2 \cdot 2 = \frac{25}{2} - 2 = \frac{21}{2}.$$

Find

P_2. : No geometric ideas. $\longrightarrow$ $\boxed{\text{Riemann Sums}}$

$f(x) = x^2$. $\quad a = 2$, $b = 5$

$\Delta_x = \frac{5-2}{n} = \frac{3}{n}$, $x_k = a + k\Delta_x = 2 + \frac{3k}{n}$

Let us ~~we~~ use right ~~left~~ Riemann Sums

$$\Delta_x \sum_{k=1}^{n} f(x_k^*) = \Delta_x \sum_{k=1}^{n} f(x_k).$$

$$= \frac{3}{n} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2$$

$$\cancel{= \frac{3}{n}\left(\sum_{k=1}^{n} (2)^2 + \sum_{k=1}^{n} \left(\frac{3k}{n}\right)^2 + 2\right.}$$

$$= \frac{3}{n} \sum_{k=1}^{n} \left(4 + \frac{9k^2}{n^2} + 2 \cdot 2 \cdot \frac{3k}{n}\right)$$

$$= \frac{3}{n}\left(\sum_{k=1}^{n} 4 + \sum_{k=1}^{n} \frac{9k^2}{n^2} + \sum_{k=1}^{n} \frac{12k}{n}\right) \left[\text{By } 2\right]$$

$$= \frac{3}{n}\left(4n + \frac{9}{n^2} \sum_{k=1}^{n} k^2 + \frac{12}{n} \sum_{k=1}^{n} k\right) \left[\text{By } 1\right]$$

$$\cancel{= \frac{3}{n}\left(4n + \frac{9}{n^2}\frac{n(n+1)}{2} + \frac{12}{n}\right.}$$

$$= \frac{3}{n}\left(4n + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{12}{n}\cdot\frac{n(n+1)}{2}\right) \quad \left[\text{By } 4 \text{ and } 5\right]$$

$$\cancel{= 12 + \frac{27}{6}\frac{n(n+1)(2n+1)}{n^3}}$$

$$= \frac{3}{\cancel{n}}\cdot 4\cancel{n} + \frac{3}{\cancel{n}}\frac{9}{n^2}\frac{\cancel{n}(n+1)(2n+1)}{6} + \frac{3}{\cancel{n}}\cdot\frac{12}{n}\frac{\cancel{n}(n+1)}{2}$$

$$= 12 + \frac{27}{6}\frac{(n+1)(2n+1)}{n^2} + \frac{36}{2}\left(\frac{n+1}{n}\right)$$

$$= 12 + \frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 18\left(1+\frac{1}{n}\right)$$

$$\text{Now } P_2 = \int_2^5 x^2\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(x_k)$$

$$= \lim_{n\to\infty}\left(12 + \frac{9}{2}\left(1+\overset{0}{\cancel{\tfrac{1}{n}}}\right)\left(2+\overset{0}{\cancel{\tfrac{1}{n}}}\right) + 18\left(1+\overset{0}{\cancel{\tfrac{1}{n}}}\right)\right)$$

$$= 12 + \frac{9}{2}(1)(2) + 18(1)$$

$$\therefore \int_2^5 (5x + x^2)\,dx = 5P_1 + P_2$$

$$= 5\cdot\frac{21}{2} + 12 + 9 + 18.$$

Now the reverse question:

~~What integral does~~

Represent

$$\lim_{n\to\infty} \left(\frac{y-1}{n}\right)\left[\frac{1}{6} + \frac{1}{6+\left(\frac{y-1}{n}\right)^5} + \frac{1}{6+\left(2\cdot\frac{y-1}{n}\right)^5} + \cdots + \frac{1}{6+\left(\frac{(n-1)(y-1)}{n}\right)^5}\right] \quad -(*1)$$

as an integral.

Compare with.

$$\lim_{n\to\infty} \Delta_x \sum_{k=1}^{n} f(x_k^*)$$

$$\Delta_x = \frac{y-1}{n} \quad ; f(x_k^*) = \frac{1}{6+\left(\boxed{\frac{(k-1)(y-1)}{n}}\right)^5}$$

~~Left Riemann Sum~~ $\Delta_x \sum_{k=1}^{n} f(x_{k-1}) = \Delta_x \sum_{k=1}^{n}$ ~~b~~

So $x_k^* = \frac{(k-1)(y-1)}{n}$.

Left Riemann Sum: $x_k^* = \cancel{a+(k-1)\Delta_x}\ x_{k-1}$

$= a + (k-1)\Delta_x$

$= a + (k-1)\left(\frac{y-1}{n}\right)$ ✓

This matches the form given so we do not have to compare it with right Riemann sum or ~~with~~ mid-point Riemann Sum.

$\therefore \quad a = 0.$

$b = a + n\Delta_x$

$= \cancel{a+} \; 0 + \cancel{n}\dfrac{(y-1)}{\cancel{n}} = y-1.$

~~Also,~~ $\cancel{f(x)}$ ~~$f(x_{k-1}) = f\left(\frac{y-1}{n}\right) = \frac{1}{\quad}$~~

Also $f(x_{k-1}) = f\left(\boxed{\dfrac{(k-1)(y-1)}{n}}\right) = \dfrac{1}{6 + \left(\boxed{\dfrac{(k-1)(y-1)}{n}}\right)^5}$

The function is $f(x) = \dfrac{1}{6 + x^5}.$

~~The Rie~~..

(*) Represents the limit of Left Riemann Sum for $f(x) = \dfrac{1}{6+x^5}$ ~~and~~ from $a = 0$ to ~~$b = y-1$~~

$b = y-1.$, that is, equal to,

$$\int_0^{y-1} \frac{1}{6+x^5}\,dx.$$