1

(3) $\sum_{k=1}^{\infty} \cancel{x^{2n}}$

3rd April)

$\cancel{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x}{}}$

Power Series (Approximation of functions). 1

Last Class $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots\ldots$?

$= \boxed{\sum_{k=0}^{\infty} \boxed{\frac{x^n}{n!}}}$

→ Each term depends on x.

→ If you add them all up we get e^x.

Suppose the function f is given. (the main foundation of differential calculus)

(James Gregory & Brook Taylor) ⇝

The Taylor series of f centred at a

$$= \sum_{k=0}^{\infty} \frac{\boxed{f^{(k)}(a)}}{k!} (x-a)^k.$$

→ kth derivative of f at a.

If $a = 0$ then it is called the Maclaurin Series.

Examples: e^x about $a=0$. (→ Maclaurin Series).

$\frac{d^k}{dx^k}(e^x) = e^x$

$f(x) = e^x$.

$\cancel{d^k} f^{(k)}(x) = e^x$

$f^{(k)}(0) = e^0 = 1$

Then Maclaurin series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

2.

~~Cos x~~. Taylor series of ~~ln(1+x) at~~ ln(x) about $x = 1$.

$f(x) = \ln(x)$, $f^{(1)}(x) = \frac{1}{x}$, ... $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$

$f(1) = \ln(1) = 0$.

$f^{(1)}(1) = 1$

$f^{(2)}(1) = -\frac{1}{1^2} = -1$

$f^{(3)}(1) = \frac{2}{1^3} = 2!$

$f^{(4)}(1) = -\frac{3.2}{1^4} = -3!$

$\vdots$

$f^{(n)}(1) = (-1)^{n-1}(n-1)!$

$f^{(2)}(x) = \frac{d}{dx}(f(x))$

$= \left(\frac{1}{x}\right)'$

$= -\frac{1}{x^2}$

$f^{(3)}(x) = \frac{2}{x^3}$.

$f^{(4)}(x) = 2\left(-\frac{3}{x^4}\right)$

$= -\frac{3.2}{x^4}$

$f^{(5)}(x) = \cancel{+}3.2\left(\frac{4}{x^5}\right)$

$= \frac{4!}{x^5}$

$\therefore$ Taylor series is $\sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!}(x-1)^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\,\cancel{(k-1)(k-2)\cdots 1}}{k\,\cancel{(k-1)\cdots 1}}(x-1)^k.$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x-1)^k}{k}.$$

$$= (x-1) - \frac{(x-1)}{2} + \frac{(x-1)^2}{3} - \frac{(x-1)^3}{4}$$

— — — —

For what values of x is this series valid?
When does it converge?

$$\underbrace{\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k}}_{a_k}$$

<u>Ratio test</u> : $\sum_{k=1}^{\infty} a_k$ converges if

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1.$$

(Might be true even if $=1$).

$$\boxed{\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|} = \lim_{k\to\infty} \left| (-1)^k \frac{(x-1)^{k+1}}{k+1} \Big/ \frac{\cancel{(-1)^{k+1}} \cancel{(x-1)^k}}{k} \right|$$

$$= \lim_{k\to\infty} \left| - \frac{(x-1)\ k}{k+1} \right|$$

$(x-1) \longrightarrow$ does not depend on k.

$$= |x-1| \lim_{k\to\infty} \left| \frac{k}{k+1} \right|$$

$$= \boxed{|x-1|}$$

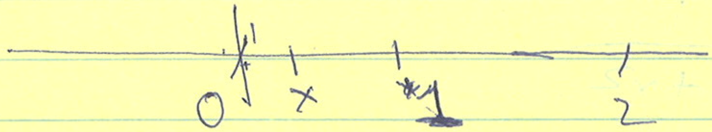
By ratio test if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$ then the series converges.

that is, $|x-1|$

$|x-1| \longrightarrow$ distance between x and 1

$\therefore \quad |x-1| < 1$

$\Rightarrow 1-1 < x < 1+1 \Rightarrow 0 < x < 2.$



A series of the form $\sum_{k=0}^{\infty} \underbrace{c_k}_{\text{numbers}} \overbrace{(x-a)^k}^{\text{numbers}}$.

are called power series with centre a.

By ratio test, we obtain ~~an an~~ a open interval around a, where the series converges, the radius of which is called radius of convergence.

$\therefore$ radius of convergence of $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x-1)^k}{k}$

$$\boxed{= 1}$$

• Ex. Find the radius of convergence of the Taylor series expansion of $\left(\frac{1}{1+x^2}\right)$ about 0.

(Maclaurin Series).

Solution:

How to find Taylor series of $\frac{1}{1+x^2}$?

We know $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ (Exercise using Taylor Series)

I would like replace x by $(-x^2)$ on both sides.

On left hand side.

$$\frac{1}{1-(-x^2)} = \frac{1}{1+x^2}.$$

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On right hand side

$\cancel{1 + (-x^2)}$

$$1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} \boxed{x}^k.$$

becomes $\displaystyle\sum_{k=0}^{\infty} (-x^2)^k$

$$= \sum_{k=0}^{\infty} \boxed{(-1)^k \, x^{2k}} = a_k.$$

Taylor series centered at 0.

Radius of convergence?

Use Ratio Test $\displaystyle\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1$

$$\Rightarrow \lim_{k\to\infty} \left| \frac{(-1)^{k+1} x^{2(k+1)}}{\cancel{(-1)^k} x^{2k}} \right|$$

$$\lim_{k\to\infty} \left| - \frac{x^{2k+2}}{\cancel{x^{2k}}} \right| = \lim_{k\to\infty} |x^2|.$$

$\Rightarrow \cancel{|x|^2} \; |x^2| < 1.$

$\Rightarrow \quad x^2 < 1$

$\Rightarrow \quad -1 < x < 1.$

Radius of convergence $= 1$.

Find the taylor series of $x^{10}\{\cos(x^{15}) + \sin(x^{15})\}$ about 0. and its radius of convergence.

Solution:

Looks very complicated.

First $f(x) = \cos(x)$	$f(0) = 1$	$g(x) = \sin x$.	$g(0) = 0$
$f^{(1)}(x) = -\sin(x)$	$f^{(1)}(0) = -\sin(0) = 0$	$g^{(1)}(x) = \cos(x)$	$g^{(1)}(0) = 1$
$f^{(2)}(x) = -\cos(x)$	$f^{(2)}(0) = -\cos(0) = -1$	$g^{(2)}(x) = -\sin(x)$	$g^{(2)}(0) = 0$
$f^{(3)}(x) = \sin(x)$	$f^{(3)}(0) = \sin(0) = 0.$	$g^{(3)}(x) = -\cos(x)$	$g^{(3)}(0) = -1$
$f^{(4)}(x) = \cos(x)$	$f^{(4)}(0) = \cos(0) = 1$	$g^{(4)}(x) = \sin(x)$	$g^{(4)}(0) = 1$
	$f^{(5)}(0) = 0.$		
	$f^{(6)}(0) = -1$		
	$f^{(7)}(0) = 0$		
	$f^{(8)}(0) = 1$		

~~Tay~~ Maclaurin series of $\cos x$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)\, x^k}{k!} = \overset{=1}{f(0)} + \frac{\overset{0}{f^{(1)}(0)}\, x}{1!} + \frac{\overset{-1}{f^{(2)}(0)}\, x}{2!} \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

Maclaurin series of ~~$\cos(x^{15})$~~ $\sin(x)$

$$= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)\, x^k}{k!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - - - -$$

$\therefore$ $(\sin(x) + \cos(x))$ Maclaurin series is.

$$1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - - - - - -$$

Maclaurin Series of $\cos(x^{15}) + \sin(x^{15})$.

$$1 + x^{15} - \frac{(x^{15})^2}{2!} - \frac{(x^{15})^3}{3!} + \frac{(x^{15})^4}{4!} + \frac{(x^{15})^5}{5!} - \cdots$$

(Replaced x by x^{15} in (*))

Maclaurin series of $x^{10}\{\cos(x^{15}) + \sin(x^{15})\}$.

$$x^{10}\left\{1 + x^{15} - \frac{(x^{15})^2}{2!} - \frac{(x^{15})^3}{3!} + \frac{(x^{15})^4}{4!} + \frac{(x^{15})^5}{5!} - \cdots\right\}$$

$$= \underbrace{x^{10}}_{a_0} + \overbrace{x^{10+15}}^{a_1} - \overbrace{\frac{x^{10+2.15}}{2!}}^{a_2} - \frac{x^{10+3.15}}{3!} = a_3$$

$$+ \underbrace{\frac{x^{10+4.15}}{4!}}_{a_4} + \underbrace{\frac{x^{10+5.15}}{5!}}_{a_5} - \cdots$$

$$|a_n| = \left|\frac{x^{10+15n}}{n!}\right|.$$

By ratiotest,

Σa_n converges for x such that

$$\boxed{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1}$$

$$\lim_{n\to\infty}\left|\frac{\frac{x^{\{10+15(n+1)\}}}{(n+1)!}}{\frac{x^{\{10+15n\}}}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{\cancel{10}+\cancel{15}n+15}}{x^{\cancel{10}+\cancel{15}n}}\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|x^{15}\frac{\cancel{n(n-1)\cdots 1}}{(n+1)\cancel{n(n-1)\cdots 1}}\right| = \lim_{n\to\infty}\left|\frac{\overbrace{x^{15}}^{\text{independent of } n}}{n}\right|$$

$= 0$.

Thus for all values of x

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1.$$

$i = \sqrt{-1}$. Prove: $e^{ix} = \cos x + i \sin x$. (Euler).

$\therefore$ ~~It~~ $\sum_{n=0}^{\infty} a_n$ converges for all x.

$\therefore$ radius of convergence is ∞.

~~Thus we can add Taylor Series (or Power~~

~~Thus if we~~

If $\sum_{k=0}^{\infty} a_k (x-a)^k$, ~~$\sum_{k=1}^{\infty} b_k (x-a)^k$ are~~ is the Taylor Series of f and $\sum_{k=0}^{\infty} b_k (x-a)^k$ is the Taylor Series for g ~~then~~ about a then.

Taylor series of $(x-a)^n f(x)$ is

$$(x-a)^n \left(\sum_{k=0}^{\infty} a_k (x-a)^k \right)$$

$$= \sum_{k=0}^{\infty} a_k (x-a)^{k+n}$$

Taylor Series of $f(x) + g(x)$ is

$$\sum_{k=0}^{\infty} (a_k + b_k)(x-a)^k.$$