Fundamental Theorems of Calculus

4th February

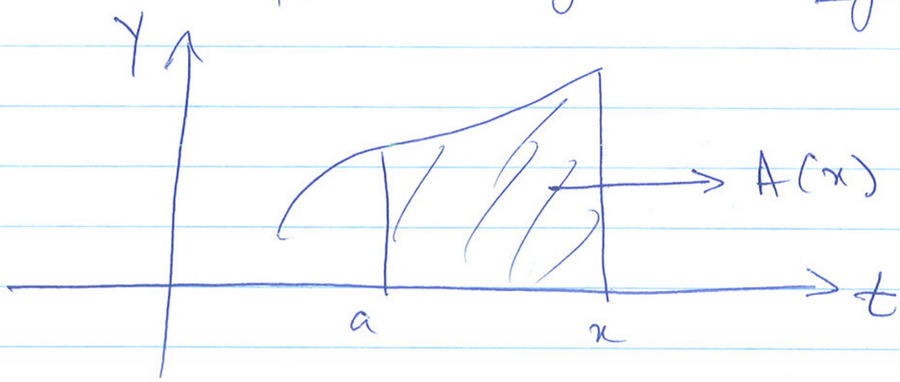
Area Functions

The area function for f with left endpoint a is defined as.

$$A(x) = \int_a^x f(t)\,dt$$

Note that we x in this place. The variable name does not matter.

The area function gives the signed area.



Qn Let $f(x) = \begin{cases} 2x+3 & x < 1 \\ 5 & x \geq 1 \end{cases}$

Compute the area function $A(x)$ starting at $a = 0$.

Answer: Since f is defined in two parts we have to split the function into 2 parts. as.

well.

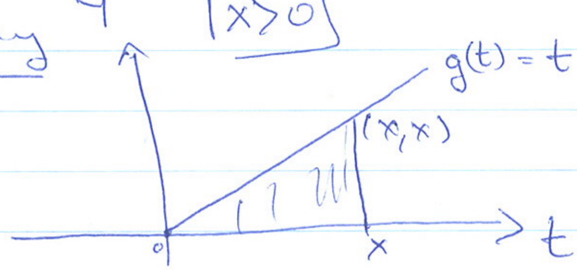
If $x \leq 1$.

$$A(x) = \int_0^x f(t)\,dt = \int_0^x (2t+3)\,dt = 2\int_0^x t\,dt + \int_0^x 3\,dt$$

$$= \underbrace{2\int_0^x t\,dx}_{P_1} + 3(x-0)$$

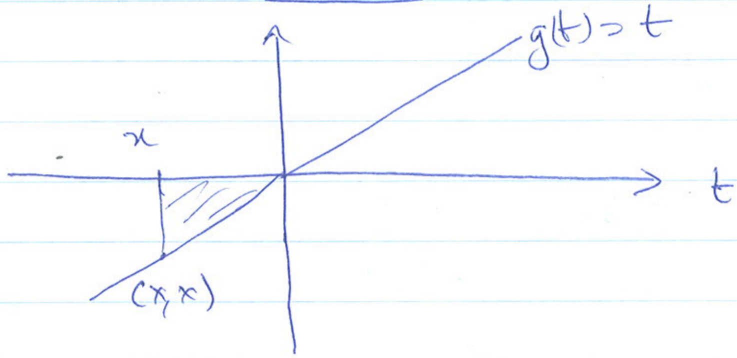
Using Geometry

$\boxed{x > 0}$



P_1 = shaded area $= \frac{1}{2}x\,x = \frac{1}{2}x^2$

$\boxed{x \leq 0}$



$$P_1 = \int_0^x t\,dt = -\int_x^0 t\,dt$$

$= -(-\text{ Shaded Area})$ [Signed Area]

$= \frac{1}{2}|x|\,|x| = \frac{1}{2}|x|^2 = \frac{1}{2}x^2$

$\therefore$ $P_1 = \frac{1}{2}x^2$ for $x < 1$.

Using Riemann Sums

$a = 0$ $\quad f(t) = t$

$b = x$ $\quad t_k = a + k\Delta_t$

$\Delta_t = \frac{x-0}{n} = \frac{x}{n}$ $\quad = \frac{kx}{n}$

x replaced by t to avoid confusion.

$$P_1 = \int_0^x t\,dt = \lim_{n\to\infty} \Delta_t \sum_{k=1}^{n} f(t_k)$$

$$= \lim_{n\to\infty} \frac{x}{n} \sum_{k=1}^{n} \left(\frac{kx}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{x}{n}\right)^2 \sum_{k=1}^{n} k.$$

Check $\frac{x^2}{2}$.

If $x \le 1$, $\quad A(x) = 2 P_1 + 3x = 2\frac{x^2}{2} + 3x = x^2 + 3x$

$A(1) = 4$

If $x \geq 1$ $\quad A(x) = \int_0^x f(x)\,dx = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt$

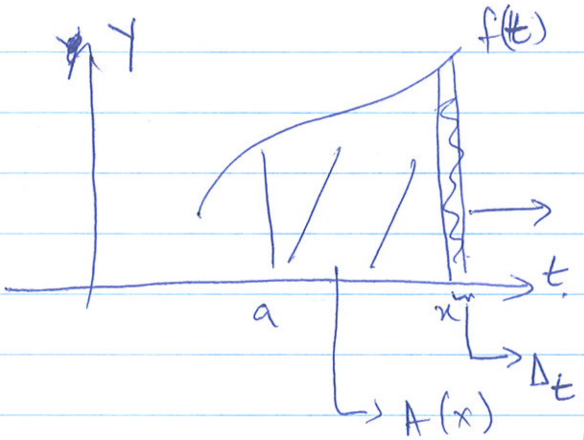
$$= A(1) + \int_1^x 5\,dt$$

$$= \cancel{A(1)}\ 4 + 5(x-1) = 5x - 1$$

Thus $A(x) = \begin{cases} x^2 + 3x & x \le 1 \\ 5x - 1 & x \geq 1 \end{cases}$

Note $f(x)$ was not differentiable

but $A(x)$ is.



Change x by Δ_t.

Then change in area.

$= A(x+\Delta_t) - A(x)$

$\doteq A(x) + f(x)\Delta_t$

What is a good guess for $A'(x)$?

Newton and Leibniz (Look at Principia)

Fundamental Theorem of Calculus - I (FTOC-I)

If f is continuous on $[a,b]$, then the area function $A(x) = \int_a^x f(t)\,dt$ satisfies

$$A'(x) = f(x)$$

Antiderivatives A function F is an antiderivative of f if $\boxed{F'(x) = f(x)}$.

Note if $F'(x) = f(x)$

then $(F(x)+10)' = F'(x) = f(x)$

rather $(F(x)+c)' = F'(x) = f(x)$ for any real number c.

Thus in general, if $F(x)$ is an ~~an~~ antiderivative of $f(x)$ then $F(x)+c$ is also an antiderivative of $f(x)$ where c can any value, e.g. $0, 1, 273, \pi$ . . . . . .

(Fundamental Theorem of Calculus - II) (FTOC - II).

If f is continuous on $[a,b]$ and F is any antiderivative of f, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b}$$ — short hand notation

- Sort of converse for FTOC - I.

keeping the previous discussion in mind, The indefinite integral of f denoted by

$\int f(x)\,dx$ = is the family of ~~anti~~ all antiderivatives of f, that is,

$$\boxed{\int f(x)\,dx} = \boxed{F(x)} + \boxed{C}$$

Indefinite integral.

Some antiderivative

any constant

Strictly speaking, the ~~antiderivative~~ indefinite integral is not a function but ~~the~~ a set of functions, where each element depends on c.

Look at the list and use FTOC-II

to compute. . $\rightarrow$ $\displaystyle\int_1^4 \left(\frac{x^2}{3}+5\right)dx = \int_1^4 \frac{x^2}{3}\,dx + \int_1^4 5\,dx$

$$= \frac{1}{3}\int_1^4 x^2\,dx + 5(4-1)$$

$$\cancel{= \frac{1}{3}\left(\frac{x^3}{3}\right)\Big|}$$

$$\cancel{= \frac{1}{3}\,\frac{x^3}{3}\Big|}$$

(Rule 2) $\displaystyle = \frac{1}{3}\left(\frac{x^{2+1}}{2+1}\Big|_1^4\right) + 15$

$$= \frac{1}{3}\left(\frac{4^3}{3} - \frac{1}{3}\right) + 15$$

~~real number~~ c.

~~Thus in general, if~~

2) $\int_2^4 \frac{1}{\sqrt[3]{x}}\,dx = \int_2^4 x^{-\frac{1}{3}}\,dx = \left.\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}\right|_2^4$ (Rule 2)

$= \frac{1}{2/3}\left(4^{2/3} - 2^{2/3}\right)$

3) $\int_0^{\frac{\pi}{4}} (\cos\theta + \sec^2\theta)\,d\theta = \int_0^{\pi/4} \cos\theta\,d\theta + \int_0^{\pi/4} \sec^2\theta\,d\theta$

$= \sin\theta\Big|_0^{\pi/4} + \tan\theta\Big|_0^{\pi/4}$

$= \left(\sin\frac{\pi}{4} - \sin 0\right) + \left(\tan\frac{\pi}{4} - \tan 0\right)$

$= \left(\frac{1}{\sqrt{2}} - 0\right) + (1 - 0) = \frac{1}{\sqrt{2}} + 1.$

Indefinite Integral: – The variable is important

$\int \frac{1}{x}\,dx = \ln|x| + c.$

$\int \frac{1}{y}\,dy = \ln|y| + c$

$\int \frac{1}{cow}\,d(cow) = \ln|cow| + c$

$\int \frac{1}{cabin}\,d(cabin) = \ln|cabin| + c$

Howeve $\int \frac{1}{2x}\,dx \neq \ln|2x| + c$ X

Definite Integral	Indefinite Integrals
• Upper (right end point) and lower ((left end point) ~~present~~ limit present.	Upper and lower limit not present.
• It is a family of function.	It is a number.
• There is a c	There is no c.

• Application of FTOC – I & II

• Suppose $G(x) = \int_1 x^2\,dx$ for all x. What is $G'(x) = ?$

~~Got~~ Solution: $G(x)$ is a number, fixed for all x

$$\text{So } G'(x) = 0$$

• Suppose $G(x) = \int_9^7 x^2\,dx$ for all x. What is $G'(x) = ?$

Solution: $G'(x) = 0$.

• Suppose $G(x) = \int_{x^2}^{x^3} e^{t^2}\,dt$ What is $G'(x)$?

Solution: By FTOC – II, if $F(x)$ is

an antiderivative of $f(x) = e^{x^2}$

then $G(x) = \int_{x^2}^{x^3} e^{t^2}\,dt = F(x^3) - F(x^2)$.

Then. $G'(x) = \frac{d}{dx}(F(x^3)) - \frac{d}{dx}(F(x^2))$

$= F'(x^3)\frac{d}{dx}(x^3) - F'(x^2)\frac{d}{dx}(x^2)$

[By Chain Rule]

$= f(x^3)(3x^2) - f(x^2)\,2x$

[Sine F is an antiderivative]

$= 3x^2 e^{(x^3)^2} - 2x\, e^{(x^2)^2}$

$= 3x^2 e^{x^6} - 2x\, e^{x^4}$.

Q] Suppose $G(x) = \int_0^{\tan x} \sqrt{\sin t}\, dt$. What is $G'(x)$?

Solution: Let $f(x) = \sqrt{\sin x}$ and its antiderivative be $F(x)$ $(F'(x) = f(x))$.

Then $G(x) = \int_0^{\tan x} \sqrt{\sin t}\, dt = F(\tan x) - F(0)$ [By FTOC - II]

$G'(x) = F'(\tan x)\frac{d}{dx}(\tan x)$ [F(0) is a number]

$= f(\tan x)\sec^2 x$ [F was an antiderivative]

$= \sqrt{\sin(\tan x)}\ \sec^2 x$.