March 4, 2014

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Approximation Techniques

Last week.

Improper Integrals with unbounded integrand.

$$\int_{-1}^{1} \frac{1}{y^2}\,dy \text{ diverges.}$$

$$\int_{0}^{1} \frac{1}{\sqrt{y}}\,dy \text{ converges.}$$

$$\int_{-1}^{1} \frac{1}{y^2}\,dy = \lim_{c\to 0^+} \int_{c}^{1} \frac{1}{y^2}\,dy + \lim_{c\to 0^-} \int_{-1}^{c} \frac{1}{y^2}\,dy$$

(Last Class) $\longrightarrow \infty$.

Infinite Intervals

$\int_{1}^{\infty} \frac{1}{x^2}\,dx$.

→ 1) f is continuous on $[a,\infty)$ then

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$ provided limit exists.

2) f is continuous on $(-\infty, b]$ then.

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to -\infty} \int_{a}^{b} f(x)\,dx$$ given limit exists.

3) If f is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to -\infty} \int_{a}^{\boxed{0}} f(x)\,dx + \lim_{b\to\infty} \int_{\boxed{0}}^{b} f(x)\,dx.$$

$\boxed{0}$ → can be replaced by any real number c.

<u>Examples</u>:

$$\int_{0}^{\infty} \sin\theta\,d\theta = \lim_{b\to\infty} \int_{0}^{b} \sin\theta\,d\theta$$

$$= \lim_{b\to\infty} -\cos\theta\Big|_{0}^{b} =$$

$$= \lim_{b\to\infty} -\cancel{\cot}\cos(b) + \cos 0$$

$$= -\lim_{b\to\infty} \cos(b) + 1$$

→ This limit does not exist

Thus $\int_{0}^{\infty} \sin\theta\,d\theta$ diverges.

$$\int_{-\infty}^{\infty} t\, e^{-t^2}\, dt$$

Substituting

$$\int_{a}^{0} t\, e^{-t^2}\, dt = -\int_{-a^2}^{0} e^{u}\, \frac{du}{2}$$

$-t^2 = u$

$-2t\,dt = du$

$\Rightarrow t\,dt = -\frac{du}{2}$

$t = 0 \quad u = 0$

$t = a \quad u = -a^2$

$t = b \quad u = -b^2$

$$= -\frac{1}{2}\left(e^{0} - e^{-a^2}\right) \cancel{= -\left(e^{a^2} + e^{0}\right)}$$

$$= \frac{1}{2}\left(e^{-a^2} - 1\right) \cancel{= -e^{a^2} + 1}$$

Substituting

Similarly

$$\int_{0}^{b} t\, e^{-t^2}\, dt = -\int_{0}^{-b^2} e^{u}\, \frac{du}{2}$$

$$= -\frac{1}{2}\left(e^{-b^2} - 1\right) = \frac{1}{2}\left(1 - e^{-b^2}\right)$$

Now.

$$\int_{-\infty}^{\infty} t\, e^{-t^2}\, dt = \lim_{a \to -\infty} \int_{a}^{0} t\, e^{-t^2}\, dt + \lim_{b \to \infty} \int_{0}^{b} t\, e^{-t^2}\, dt$$

$$= \lim_{a \to -\infty} \frac{1}{2}\left(e^{-a^2} - 1\right) + \lim_{b \to \infty} \frac{1}{2}\left(1 - e^{-b^2}\right)$$

$$= \lim_{a \to -\infty} \frac{1}{2}(e^{-a^2}) - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}}$$

$$- \lim_{b \to \infty} \frac{1}{2}(e^{-b^2})$$

$\cancel{a^2} \rightarrow \cancel{\infty}$
$-a^2$

$a^2 \longrightarrow \infty \Rightarrow -a^2 \longrightarrow -\infty \Rightarrow e^{-a^2} \longrightarrow 0$.

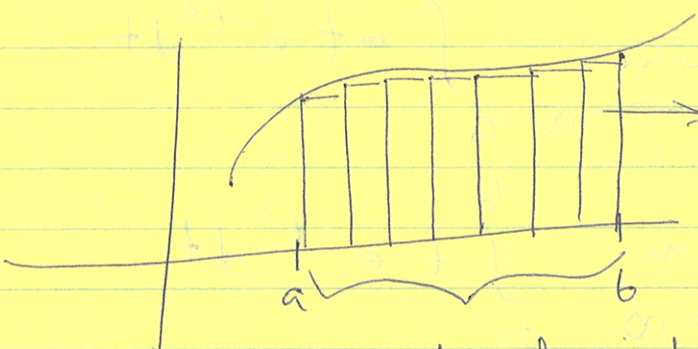
$b^2 \longrightarrow \infty \Rightarrow -b^2 \longrightarrow -\infty \Rightarrow e^{-b^2} \longrightarrow 0$.

$$= 0.$$

$$\int_{-\infty}^{\infty} t\, e^{-t^2}\, dt = 0.$$

<u>Numerical</u> <u>Integrations</u>

Full circle back to Riemann Sums.



$\rightarrow$ approximate each chunk by rectangles.

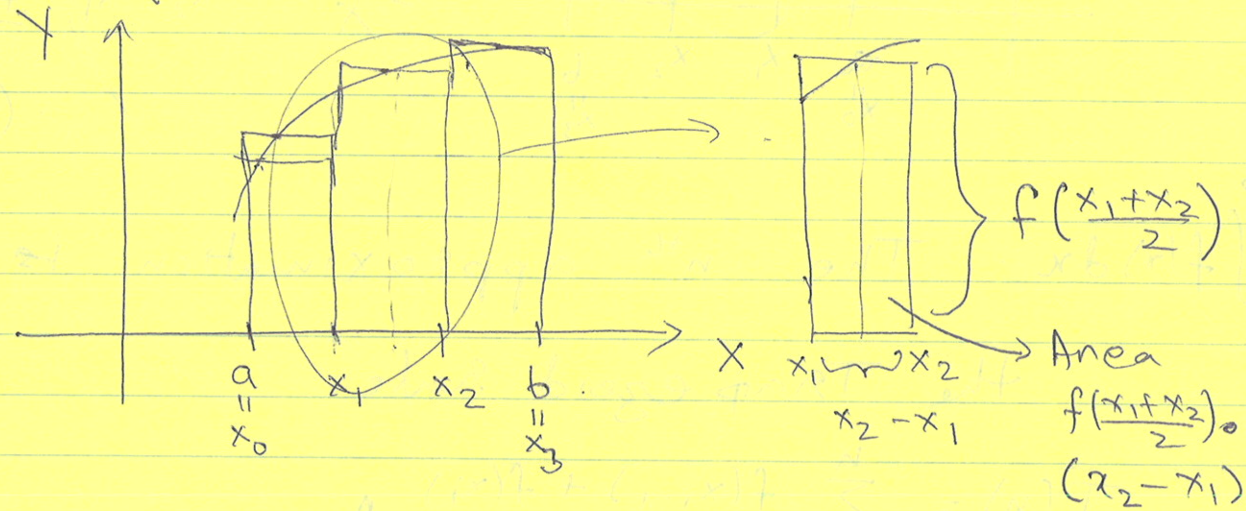
divide into chunks.

There are many function we cannot integrate

$\int e^{-x^2}\,dx \qquad \int \sqrt{\sin x}\,dx$. Best way out $\downarrow$ $\boxed{\text{Approximate}}$

Three Techniques

1) Midpoint Rule (Midpoint Riemann Sums)



$\int_a^b f(x)\,dx$. The n^{th} approximation by midpoint rule. is.

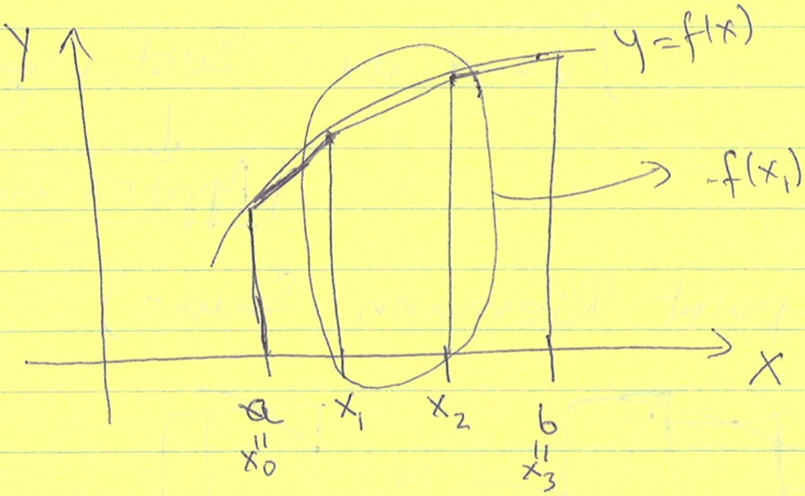
$$M(n) = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x$$

$$= \Delta_x\left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$$

where $\Delta_x = \frac{b-a}{n}$ and

$x_i = a + i\Delta_x$ for $i = 0, \ldots n$.

~~Examples~~: Trapezoid rule



Area is $\frac{f(x_1)+f(x_2)}{2} \cdot (x_2 - x_1)$

$\int_a^b f(x)\,dx$. The n^{th} approximation ~~is~~ by the Trapezoid rule.

$$T(n) = \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x.$$

All others 1

$$= \Delta_x \left[\frac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n) \right]$$

where $\Delta_x = \frac{b-a}{n}$ and $x_i = a + i\Delta_x$ for $i = 0, \ldots n$.

Error: $\left| \left(\int_a^b f(x)\,dx \right) - (\text{Approximation}) \right|$ (Absolute error)

$$\frac{\left| \left(\int_a^b f(x)\,dx \right) - (\text{Approximation}) \right|}{\left(\int_a^b f(x)\,dx \right)}$$ (Relative error)

Approximate

<u>Example</u>: $\int_0^1 x^3 dx$ by Trapezoid rule with $n = 3$. Find the relative error.

- ~~Td~~ $\Delta x = \frac{1-0}{3} = \frac{1}{3}$. $\cancel{\left(\frac{b-a}{n}\right)}$ $\left(\Delta x = \frac{b-a}{n}\right)$

$x_0 = 0$; $x_1 = 0 + \frac{1}{3} = \frac{1}{3}$; $x_2 = 0 + 2 \cdot \frac{1}{3} = \frac{2}{3}$

$x_3 = 0 + 3 \cdot \frac{1}{3} = 1$ $\quad \left[x_i = a + i\,\Delta x \right]$

$$T(3) = \Delta x \left[\tfrac{1}{2} f(x_0) + f(x_1) + \cdots f(x_{n-1}) + \tfrac{1}{2} f(x_n) \right]$$

$$= \Delta x \left[\tfrac{1}{2} f(x_0) + f(x_1) + f(x_2) + \tfrac{1}{2} f(x_3) \right]$$

$$= \frac{1}{3} \left[\frac{1}{2} (x_0)^3 + x_1^3 + x_2^3 + \frac{1}{2} x_3^3 \right]$$

$$= \frac{1}{3} \left[\frac{1}{2} (0)^3 + \left(\frac{1}{3}\right)^3 + \left(\frac{2}{3}\right)^3 + \frac{1}{2} (1)^3 \right]$$

$$= \frac{1}{3} \left[\frac{1}{27} + \frac{8}{27} + \frac{1}{2} \right] = \frac{1}{3} \left[\frac{1}{3} + \frac{1}{2} \right]$$

$$= \frac{1}{3} \cdot \frac{5}{6} = \boxed{\frac{5}{18}}$$

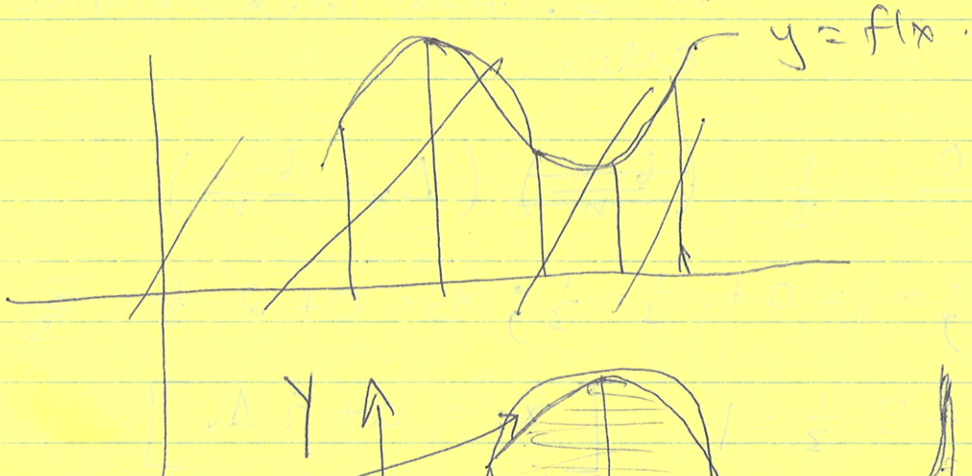
$$\int_0^1 x^3 dx = \left. \frac{x^4}{4} \right|_0^1 = \frac{1}{4}.$$

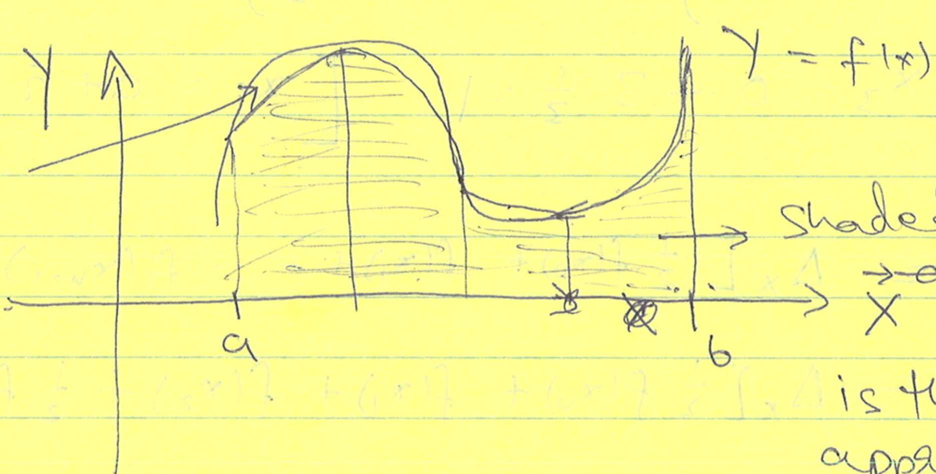
Approximation

$$\text{Relative error} = \left| \frac{\int_0^1 x^3 dx - T(3)}{\int_0^1 x^3 dx} \right| = \frac{\left| \frac{1}{4} - \frac{5}{18} \right|}{\frac{1}{4}} = \boxed{\frac{1}{9}}$$

(Kepler, Simpson → Divinity
Astronomy)

3) Simpsons Rule — piecewise quadratic approximation





unshaded area is the error

shaded area → ~~approximation~~ is the approximation.

$\boxed{n \text{ is even}}$ $\qquad \int_a^b f(x)\,dx.$

n^{th} approximation is by Simpson's Rule

is $$S(n) = \frac{\Delta x}{3}\Big[f(x_0) + 4(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

where n is <u>even</u> $\quad \Delta_x = \frac{b-a}{n}$.

$x_i = a + i\,\Delta_x$ for $i = 0, \ldots n.$

Coefficients are (1, 4, 2, 4, 2, ... 2, 4, 1