1.

SUBSTITUTION & INTEGRATION BY PARTS

6th February

[FROM LAST CLASS]

The Fundamental Theorem of Calculus (FTOC)

If f is continuous on $[a,b]$, then the area function $A(x) = \int_a^x f(t)\,dt$ satisfies

$$A'(x) = f(x)$$

If F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b .$$

The indefinite integral is defined as

$$\int f(x)\,dx = \underbrace{F(x)}_{\text{any antiderivative}} + \overbrace{C}^{\text{some constant}}$$

Q: Suppose $G(x) = \int_0^{\tan x} \sqrt{\sin t}\,dt$. What is $G'(x)$?

Solution: Let $f(x) = \sin x$ and $F(x)$ be its antiderivative ($F'(x) = f(x)$) Then

$$G(x) = \int_0^{\tan x} \sqrt{\sin t}\,dt = F(\tan x) - F(0)$$

[By FTOC]

$$G'(x) = F'(\tan x)\,\frac{d}{dx}(\tan x)$$

$$= f(\tan x)\,\sec^2 x.$$

$$= \sqrt{\sin(\tan x)}\ \sec^2 x.$$

~~Suppose G(x) =~~

Methodd of substitution.

- $(e^{ax})' = e^{ax}\frac{d}{dx}(ax) = a e^{ax}.$

$$\Rightarrow \left(\frac{e^{ax}}{a}\right)' = e^{ax}.$$

Thus $\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$

$(\sin(ax))' = a\cos(ax)$

$\Rightarrow \int (\cos(ax))\,dx = \frac{\sin(ax)}{a} + C$ for all a

Similarly $\int \sin(ax)\,dx = -\frac{\cos(ax)}{a} + C$

¶ General ~~Prin~~ Principle: Substitution Rule

Let $u = g(x)$ where $g'(x)$ is continuous.

Let f be continuous on the range of g

Then $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du.$

- $\int e^{9x}\,dx$

$u = g(x) = 9x$.

~~du =~~ $g'(x) = 9$ ⟵ We shall write. ⟶ $\boxed{du = 9dx}$ ~~du = 9 dx Replace du by 9dx~~

$\Rightarrow \frac{du}{9} = dx$.

This does not make sense mathematically. Notational

Then $\int e^{9x}\,dx = \boxed{\int e^u \frac{du}{9}} = \frac{1}{9}\int e^u\, du$

Replaced all x

$= \frac{1}{9}(e^u + c)$

$= \frac{1}{9}e^u + c$ [$\frac{c}{9}$ is just another constant]

Putting $9x$ back. ⟵ $= \boxed{\frac{1}{9}e^{9x} + c}$

- Compute $\int(\sin(2\theta) - \cos(3\theta))\,d\theta$

$= \int \sin(2\theta)\,d\theta - \int\cos(3\theta)\,d\theta$

$u = 2\theta$

$\boxed{du = 2d\theta}$

$\frac{du}{2} = d\theta \Rightarrow \int \sin 2\theta\, d\theta = \boxed{\int \sin u\, \frac{du}{2}} = \frac{1}{2}(-\cos u + c)$

Replaced all θ

$= -\frac{1}{2}\cos u + c$

$= -\frac{1}{2}\cos 2\theta + c$

Check $\int \cos 3\theta\, d\theta = \frac{1}{3}\sin 3\theta + c$.

$\therefore \int(\sin(2\theta) - \cos 3\theta)\,d\theta = -\frac{1}{2}\cos 2\theta - \frac{1}{3}\sin 3\theta + c$

Substitution Rule for Definite Integral

Let $u = g(x)$ and f be continuous on the range of g. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

The Strategy

Choose u as the complicated part of the integrand. If $u = g(x)$, then replace

$g(x)$ by u

$g'(x)dx$ by du

a by $g(a)$

b by $g(b)$

- NO x is present in the expression at this stage.
- Substitute again or integrate.
- $\displaystyle\int_2^3 \frac{x^2}{x^3+2}\, dx.$

Observe. $(x^3+2)' = 3x^2$.

So let $u = x^3+2 = (g(x))$

$du = \underbrace{3x^2}_{g'(x)}\, dx.$

$g(2) = 10$, $g(3) = 29$.

$$\int_2^3 \frac{x^2}{x^3+2}\,dx.$$

$x^3+2 \longrightarrow u$

$3x^2\,dx \longrightarrow du$

$x^2\,dx \longrightarrow \frac{du}{3}$

$2 \longrightarrow g(2) = 10$

$3 \longrightarrow g(3) = 29$

gives us.

$$\int_{g(2)}^{g(3)} \frac{du}{3u} = \frac{1}{3}\int_{10}^{29} \frac{du}{u} = \frac{1}{3}\left(\ln(29) - \ln(10)\right)$$

$$\int_1^2 2x^5\sqrt{x^2-1}\,dx.$$

$u = g(x) = x^2-1$

$du = g'(x)\,dx = 2x\,dx.$

$g(1) = 0 \quad , g(2) = 3$

$$\int_1^2 2x^5\sqrt{x^2-1}\,dx = \int_1^2 x^4\sqrt{x^2-1}\;2x\,dx$$

What to do?

$x^2 = u+1$

$x^4 = (u+1)^2 = u^2+2u+1$

$$\therefore \int_1^2 x^4\sqrt{x^2-1}\cdot 2x\,dx = \int_0^3 (u^2+2u+1)\sqrt{u}\,du$$

$$= \frac{u^{7/2}}{7/2} + 2\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\Bigg|_0^3$$

$$= \frac{3^{7/2}}{7/2} + 2\frac{(3)^{5/2}}{5/2} + \frac{(3)^{3/2}}{3/2}$$

Method ② Try $x^2-1 = u^2 = (g(x))^2$ $\left(\begin{array}{l} \text{so } g(x) = \sqrt{x^2-1} \\ g(1) = 0 \\ g(2) = \sqrt{3} \end{array}\right)$

Then we have replace $2x\,dx$ by $u\,du$.

We get $\int_{g(1)}^{g(2)} \sqrt{u^2}\,(1+2u^2+u^4)\,u\,du$.

Integration by Parts

(*) $\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$.

$u(x) \longleftarrow$ suppress x.

Set $\boxed{u} = f(x)$, $v = g(x)$

$du = f'(x)\,dx$, $dv = g'(x)\,dx$.

(*) becomes $\int u\,dv = uv - \int v\,du$.

This corresponds to ~~integrati~~ product rule for derivatives

Indeed $(f(x)\,g(x))' = f'(x)\,g(x) + f(x)\,g'(x)$

$f(x)\,g'(x) = (f(x)\,g(x))' - f'(x)\,g(x)$.

Integrating we get *

• 1) $\int x\,e^x\,dx$ $\quad u = x \quad v(x) = e^x$

$du = dx \quad dv = e^x\,dx$.

Then $\int x\,e^x\,dx = \int u\,dv$

$= uv - \int v\,du$

$= x\,e^x - \int e^x\,dx$

$= x\,e^x - e^x + c$

2) $\int \ln x \, dx$.

$u = \ln x$. $\quad dv = dx$.

$du = \frac{1}{x} dx \quad v = \int dv = \int dx = x$

$= \int u \, dv$

$= uv - \int v \, du$

$= (\ln x)(x) - \int x \frac{1}{x} dx = x \ln x - x + c$

3) $\int e^x \sin x \, dx$.

$u = \sin x \quad dv = e^x dx$

$du = \cos x \, dx \quad v = \int e^x dx = e^x$

$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du$.

$= (\sin x) e^x - \int e^x \cos x \, dx \cdots$ (1).

$\int e^x \cos x \, dx$:

$u = \cos x \quad \cancel{dv = \cos x \, dx}$.

$du = -\sin x \, dx \quad \cancel{v = \int \cos x \, dx = \sin}$

$dv = e^x dx$

$v = \int e^x dx = e^x$.

$= \int u \, dv$

$= uv - \int v \, du$

$= (\cos x) e^x - \int e^x (-\sin x) \, dx$

$= \cos x \, e^x + \int e^x \sin x \, dx. \cdots$ (2).

Replacing (2) in (1), we get,

$\int e^x \sin x \, dx = (\sin x) e^x - (\cos(x) e^x + \int e^x \sin x \, dx)$

$= \sin x \, e^x - \cos x \, e^x - \int e^x \sin x \, dx$.

$\Rightarrow 2 \int e^x \sin x \, dx = \sin x \, e^x - \cos x \, e^x$

$\Rightarrow \int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x)$.

PREFERENCE LIST

L ogarithm ($\log(x)$)
I nverse ($\arctan(x)$, $\arcsin x$)
A lgebraic (x^2, $x^{15}+8x$)
T rignometric ($\sin(x)$, $\tan(x)$ ...)
E xponential (e^x, e^{20x}, 2^x)
D uh!!! (dx)

There are exceptions use your judgment.

For definite integrals

Let u and v be differentiable then

$$\int_a^b u(x)\, v'(x)\, dx = u(x)\, v(x)\Big|_a^b - \int_a^b v(x) u'(x) dx.$$

(4) $\int_1^2 x^2 e^x \, dx$. — $\boxed{\text{Forget about limits}}$

Choose u, v. Algebraic comes first.

$u = x^2$ $\qquad$ $v = e^x$

$du = 2x\,dx$ $\qquad$ $dv = e^x\,dx$

$$\int \underbrace{x^2}_{u} \overbrace{e^x dx}^{dv} = \underbrace{x^2}_{u}\ \underbrace{e^x}_{v} + \int \underbrace{e^x}_{v}\ \underbrace{2x\,dx}_{du}$$

$$= x^2 e^x + 2\int x\, e^x\, dx.$$

By problem 1 $\int x e^x dx = x e^x - e^x + c$

$$\int x^2 e^x dx = x^2 e^x + 2(x e^x - e^x + c) = (x^2+2x-2)e^x + c$$

$\boxed{\text{Put limits back}}$ $\int_1^2 x^2 e^x\,dx = \cancel{x^2 e^x +}$

$= (x^2+2x-2)e^x \Big|_1^2$

$= (2^2 + 2.2 - 2)e^2$
$- (1^2 + 2.1 - 2)e^1$
$= 5e^2 - e.$

5) $\int_1^2 \frac{(\ln x)^2}{(x)^2}\,dx$

$u = (\ln x)^2$ $\quad dv = \frac{1}{x^2}dx.$

$du = \frac{2(\ln x)}{x}dx$ $\quad v = \int \frac{1}{x^2}dx = \int x^{-2}dx$

$= \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1}$

$= -x^{-1}$

$\int \frac{(\ln x)^2}{x^2}dx = \int u\,dv$

$= uv - \int v\,du = (\ln(x))^2(-x^{-1})$

$- \int(-x^{-1})2\ln x\left(\frac{1}{x}\right)dx$

$= -\frac{(\ln(x))^2}{x} + 2\int \ln x\, x^{-2}\,dx.$ $\leftarrow$ (1)

$u = \ln x$ $\quad dv = x^{-2}dx$
$du = \frac{1}{x}dx$ $\quad v = \int x^{-2}dx = -x^{-1}$

$\therefore \int \frac{\ln x}{x^2}dx = \int u\,dv = uv - \int v\,du.$

$= \cancel{\int \ln x\, x^{-1} = (\ln x)(-x^{-1}) - \int(-x)}$

$= (\ln x)(-x^{-1}) - \int(-x^{-1})\frac{1}{x}dx.$

$$= -\frac{\ln x}{x} + \left(\int \frac{dx}{x^2}\right)$$

$$= -\frac{\ln x}{x} + \frac{x^{-2+1}}{-2+1} + C$$

$$= -\frac{\ln x}{x} - x^{-1} + C$$

$$= -\frac{\ln x}{x} - \frac{1}{x} + C \qquad (2).$$

~~Thus~~ Replacing it in (1), we get,

$$\int_1^2 \left(\frac{\ln x}{x}\right)^2 dx = -\frac{(\ln x)^2}{x} + 2\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Bigg|_1^2$$

Leave out the C