1

6th ~~February~~ March.

Approximating Integrals

Simpson's Rule

n- even. $\int_a^b f(x)\, dx$.

n^{th} approximation by Simpson's Rule is

$$S(n) = \boxed{\frac{\Delta x}{3}} \Big[f(x_0) + 4 f(x_1) + 2 f(x_2) + 4 f(x_3) + \dots + 2 f(x_{n-2}) + 4 f(x_{n-1}) + f(x_n) \Big]$$

where n is even $\Delta x = \frac{b-a}{n}$

$x_i = a + i \Delta x$ for $i = 0, 1 \dots n$.

$$1 . f(x_0) + 4 f(x_1) + 2 f(x_2) + \cancel{4 \dots 2 \; 4 \; 2} \dots 2 f(x_{n-2}) + 4 f(x_{n-1}) + 1 . f(x_n) \Big]$$

Coeffecients:

~~1 4 2 4~~.

[1 4 2 4 2 4 2 4 2 . . . . 4 2 4 1.]

Trapezoid rule

$[\frac{1}{2}$ 1 1 1 1 . . . . 1 1 $\frac{1}{2}]$

Example: $\int_0^1 x^3\,dx$ by Simpson's rule with $n=4$. Find the relative error.

$$\Delta x = \frac{1-0}{4} = \frac{1}{4}.$$

$$x_0 = 0\ ;\ x_1 = 0+\frac{1}{4} = \frac{1}{4}\ ;\ x_2 = 0+2\,\frac{1}{4} = \frac{1}{2}$$

$$x_3 = 0+3\,\frac{1}{4} = \frac{3}{4},\quad x_4 = 1.$$

$$S(4) = \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\Big]$$

$$= \frac{1/4}{3}\Big[(x_0)^3 + 4(x_1)^3 + 2(x_2)^3 + 4(x_3)^3 + (x_4)^3\Big]$$

$$= \frac{1}{12}\Big[\cancel{4}(0)^3 + 4\left(\tfrac{1}{4}\right)^3 + 2\left(\tfrac{1}{2}\right)^3 + 4\left(\tfrac{3}{4}\right)^3 + 1^3\Big]$$

$$= \frac{1}{4}.$$

$$\int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}.$$

$$\text{Relative error} = \frac{\left|\int_0^1 x^3\,dx - S(4)\right|}{\left|\int_0^1 x^3\,dx\right|} = 0!\,!$$

General Error Bounds

Want to find $\int_a^b f(x)\,dx$. Found by approximation. some guess (Trapezoid, Midpoint, Simpson's).

How good is the guess?

One way to check is by calculating the error. But to find it exactly you need to find $\int_a^b f(x)\,dx$ which defeats the purpose. So we find bounds.

Midpoint Rule and Trapezoid Rule

Find a number k such that

$|f''(x)| < k$ for x in $[a,b]$.

↳ 2nd Derivative.

Midpoint rule with 'n' intervals.

$$E_M \doteq \left| \int_a^b f(x)\,dx - M(n) \right| \leq \frac{k(b-a)^3}{24n^2}$$

Error in Midpoint rule.

Error in Trapezoid Rule.

$$E_T = \left| \int_a^b f(x)\,dx - T(n) \right| \le \frac{K(b-a)^3}{12 n^2}$$

Trapezoid Rule with n intervals.

Simpson's Rule. Find a number k such that $| f^{(4)}(x) | < k$ for x in $[a,b]$

4th derivative.

Error in Simpson's Rule.

$$E_S = \left| \int_a^b f(x)\,dx - S(n) \right| \le \frac{k}{180} \frac{(b-a)^5}{n^4}$$

Hardest Part $\longrightarrow$ find k.

- Find an error bound for approximation of $\int_a^b f(x)\,dx = \int_a^b x^3\,dx$ for n intervals, by the Simpson's Rule.

<u>Solution:</u> $f(x) = x^3$

$\cancel{f^{(4)}(x}$ $f'(x) = 3x^2$

$f^{(2)}(x) = 6x$; $f^{(3)}(x) = 6$; $f^{(4)}(x) = 0$.

We want k such that

$|f^{(4)}(x)| \leq k$ for all x in $[a,b]$.

Choose $\boxed{k = 0}$. It satisfies. $|f^{(4)}(x)| \leq k$ for all x

Thus the error bound is

$$E_S \leq \frac{0}{180} \frac{(b-a)^5}{n^4} = 0.$$

Thus there is no error. ☒

(2) How large should we take n so that the Simpson's Rule approximation.

to $\displaystyle\int_1^2 \frac{1}{x^3}\,dx$ is accurate to 2×10^{-12}.

Solution: $f(x)=\frac{1}{x^3}$.

$f^{(1)}(x) = -3x^{-4}$

$f^{(2)}(x) = 12x^{-5}$

$f^{(3)}(x) = -60x^{-6}$

$f^{(4)}(x) = 360x^{-7}$

$= \frac{360}{x^7}$

Find 'k

Some number such that

$\left|\frac{360}{x^7}\right| = |f^{(4)}(x)| \leq k$ for x in $\boxed{[1, 2]}$

Very important.

Now $1 \leq x \leq 2$

$\Rightarrow \frac{1}{x} \leq 1$

$\Rightarrow \frac{1}{x^7} \leq 1$

$\Rightarrow \underbrace{\frac{360}{x^7}}_{|f^4(x)|} \leq 360$

$\therefore$ We can choose $k=360$.

Thus $E_S \leq \frac{360}{180} \frac{(2-1)^5}{n^4} = \frac{2}{n^4}$ --- (1).

We want $E_S \leq 2 \times 10^{-12}$.

~~It~~ By (1), it is sufficient to show

$$\frac{\cancel{2}}{n^4} \leq \cancel{2} \times 10^{-12}.$$

$$\Rightarrow \frac{1}{n^4} \leq 10^{-12} = \frac{1}{10^{12}}$$

$$\Rightarrow 10^{12} \leq n^4$$

$$\Rightarrow (10^{12})^{\frac{1}{4}} \leq (n^4)^{\frac{1}{4}} = n$$

$$\Rightarrow 10^{12 \cdot \frac{1}{4}} \leq n$$

$$\Rightarrow 10^3 = 1000 \leq n$$

Thus the smallest possible value of n is $\underline{n = 1000}$.

Application of Integration

Theory of Probability

Random Variable : is not a variable.

It is a function.

$$X(\quad) = \text{outcome}.$$

random variable

situation

Situation is tossing of a coin

~~Xt~~ X tells us whether it is heads or tails.

Situation could be throwing a dart.

X would tell us where the dart hit the dart board.

Discrete random variable

Takes discrete value ; like 0, 1, 2
(tossing of a coin, head, tails,
throwing a dice) Jack, Queen ...

Continuous random variable

Takes continuous value ; as in an interval [0, 1] or any real value.
[The time the bus arrives, the place where the lightning strikes ... ...]

We are interested in calculating the probability of events.

probability mass function / (pmf).
probability density function (pdf)

Cumulative density function (cdf).