8th April

Taylor Series of $f$ centred at $a$

$$= \sum_{k=0}^{\infty} \underbrace{\frac{f^{(k)}(a)}{k!}}_{a_k} (x-a)^k \qquad f^{(k)}(a) \longrightarrow k^{th} \text{ derivative of } f \text{ at } a.$$

$a=0 \longrightarrow$ Maclaurin Series.

$\rightarrow$ Radius of convergence : Use the ~~root test~~

Ratio test : $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

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Compute Taylor Series of $\arctan(x)$ about $0$.

$f(x) = \arctan(x)$

$f'(x) = \frac{1}{1+x^2} \longrightarrow$ we know the Maclaurin Series of $\frac{1}{1+x^2}$.

and $\boxed{\int_0^x \frac{1}{1+y^2}\, dy = \arctan(x)}$

~~Maclaur~~

Maclaurin Series of $\frac{1}{1+y^2}$ is $\sum_{k=0}^{\infty} (-1)^k (y)^{2k}$ for $-1<y<1$.

Maclaurin Series of $\int_0^x \frac{1}{1+y^2}\, dy$ is $\sum_{k=0}^{\infty} \int_0^x (-1)^k y^{2k}\, dy$.

$$= \sum_{k=0}^{\infty} (-1)^k y^{2k+1} \Big|_0^x$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)}}{2k+1}.$$

• If Maclaurin Series of $f$ is $\sum_{k=0}^{\infty} a_k x^k$ then

~~" " " g f h(x) = g(x~~

~~Maclaurin Series of $h(x) = f(x^n)$ is $\sum_{k=0}^{\infty} a_k (x^n)$~~.

Maclaurin Series of $h(x) = f(x^n)$ is $\sum_{k=0}^{\infty} a_k (x^n)^k$.

Taylor series of $f$ is $\sum_{k=0}^{\infty} a_k (x-a)^k$,

" " " $g$ is $\sum_{k=0}^{\infty} b_k (x-a)^k$.

Then " " " $f+g$ is $\sum_{k=0}^{\infty} (a_k+b_k)(x-a)^k$.

Then " " " $f'(x)$ is $\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k (x-a)^k\right) = \sum_{k=1} a_k k (x-a)^{k-1}$

$$\int_c^d f(x)\,dx = \int_c^d \sum_{k=0}^{\infty} a_k (x-a)^k\,dx = \sum_{k=0}^{\infty} a_k \int_c^d (x-a)^k\,dx.$$

~~In Taylor series about c a~~

Thus we can treat $\sum_{k=0}^{\infty}$ almost as a finite sum.

(In general this is a very subtle issue).

Be careful about Radius of Convergence

- Compute $\sum_{k=0}^{\infty} \frac{(-1)^k 3^{-k-\frac{1}{2}}}{2k+1}$ using $\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$.

for $|x|<1$

~~Note.~~

$\boxed{\text{~~Note.~~ } x = \cancel{\;}}$

$\cancel{= \sum_{k=0}^{\infty} (-1)^k (3^{\frac{1}{2}})^k}$.

$$\cancel{\sum_{k=0}} \quad \sum_{k=0}^{\infty} \frac{(-1)^k 3^{-k-\frac{1}{2}}}{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k (3^{-\frac{1}{2}})^{2k+1}}{2k+1}.$$

Thus $x = 3^{-\frac{1}{2}}$. Therefore, $\sum_{k=0} \frac{(-1)^k 3^{-k-\frac{1}{2}}}{2k+1}$

$$= \arctan(3^{-\frac{1}{2}})$$

$$= \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

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- Compute

$\sum_{k=0}^{\infty} \frac{2^k}{(k+1)!}$ using $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ Are the same

shifted by 1

$$\sum_{k=0}^{\infty} \frac{2^k}{(k+1)!} = \sum_{k=0}^{\infty} \frac{1}{2} \frac{2^{k+1}}{(k+1)!} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{2^{k+1}}{(k+1)!}$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{2^k}{k!}$$

$$\cancel{= \frac{1}{2}\left\{ \sum_{k=0}^{\infty} 2 \right.} \qquad \cancel{= \frac{1}{2}\left\{ \sum_{k=0}^{\infty} \frac{2^k}{k!} + 1 - 1 \right\}}$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{2^k}{k!} = \frac{1}{2}\left\{\sum_{k=1}^{\infty} \frac{2^k}{k!} + \frac{2^0}{0!} - \frac{2^0}{0!}\right\} \quad [0! = 1]$$

$$= \frac{1}{2}\left\{\sum_{k=0}^{\infty} \frac{2^k}{k!} - \frac{1}{1}\right\} \quad [2^0 = 1]$$

$$= \frac{1}{2}\left\{e^2 - 1\right\} \quad \left[e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\right]$$

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Series:
- Divergence Test $\sum_{k=1}^{\infty} k$, $\sum_{k=1}^{\infty} \sin(k)$
- Telescopic / geometric / p-series $\sum_{k=1}^{\infty} \sin(\frac{1}{k}) - \sin(\frac{1}{k+1})$
- Integral Test - $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$
- Ratio Test - $\sum_{k=0}^{\infty} \frac{x^k}{k!}$,
- Comparison Test. $\sum_{k=1}^{\infty} \frac{1}{k!+1}$, $\sum_{k=0}^{\infty} \frac{k^{k+2}}{k!+k^2+2}$

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- Find ~~Maclaurin~~ Taylor Series of $\int^{x-1} \frac{t}{1-t^8}\,dt$ about $x=1$.

$$\frac{1}{1-t^8} = \sum_{k=0}^{\infty} (t^8)^k = \sum_{k=0}^{\infty} t^{8k}$$

$$\frac{t}{1-t^8} = \sum_{k=0}^{\infty} t^{8k+1}, \quad \int_0^{x-1} \frac{t}{1-t^8}\,dt = \sum_{k=0}^{\infty} \int_0^{x-1} t^{8k+1}\,dt$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(x-1)^{8k+2}}{8k+2}}$$

$$\Rightarrow \cancel{\sum_{k=0}^{\infty}}$$

Examp

~~Suppose $f(x) = \sum_{x=0}^{\infty} x^{2x}$~~

~~What is $\int_0^{1/2} f(x)\,dx$?~~

Find the radius of convergence for

$$\sum_{k=0}^{\infty} \underbrace{\frac{(-1)^k k^{2k}\, 3^{k+5}\, x^{2k}}{(6k+5)!}}_{a_k}.$$

Use $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e$ if necessary.

By ratio test the series converes if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\dfrac{(-1)^{k+1}(k+1)^{2(k+1)}\,3^{k+1+5}\,x^{2(k+1)}}{(6(k+1)+5)!}}{\dfrac{(-1)^k (k)^{2k}\,3^{k+5}\,x^{2k}}{(6k+5)!}}\right|$$

$$= \lim_{k\to\infty}\left|\underbrace{\frac{(k+1)^{2k+2}}{k^{2k}}\cdot\frac{3^{k+6}}{3^{k+5}}\cdot\frac{x^{2(k+1)}}{x^{2k}}\cdot\frac{(6k+5)!}{(6k+11)!}}\right|$$

(group terms).

$$\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{2k} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^{2k} = \lim_{k\to\infty}\left(\left(1+\frac{1}{k}\right)^k\right)^2 = e^2.$$

$$= \lim_{k\to\infty}\left|\underbrace{\cancel{\left(\frac{k+1}{k}\right)^{2k}}}_{e^2}\frac{(k+1)^2\,\overbrace{(3)\;x^2}^{\text{do not depend on } k}}{(6k+11)(6k+10)\cdots(6k+6)}\right|$$

$$= 3x^2e^2\lim_{k\to\infty}\left|\left(\frac{k+1}{6k+11}\right)\left(\frac{k+1}{6k+10}\right)\frac{1}{(6k+9)\cdots(6k+6)}\right|$$

$$= e^{2} 3x^{2} \lim_{k \to \infty} \left| \underbrace{\left(\frac{1+\frac{1}{k}}{6+\frac{11}{k}}\right)}_{\to 1/6} \underbrace{\left(\frac{1+\frac{1}{k}}{6+\frac{10}{k}}\right)}_{\to \frac{1}{6}} \underbrace{\frac{1}{(6k+9) \cdots (6k+6)}}_{\to 0} \right|$$

$$= 0.$$

$\therefore$ Radius of convergence is $\infty$.

- ~~Find Taylor series of $\int$~~
- Consider the function

$$F(x) = \begin{cases} a & \text{if } x \le 0 \\ kx^2 & \text{if } 0 < x < 1 \\ b & \text{if } x \ge 1 \end{cases}$$

Find $a$, $k$ and $b$ for which $F$ is a valid cumulative distribution function of a continuous random variable. Find its mean ~~and Variance~~.

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Solution: cd.f properties: 1> $\lim_{x \to \infty} F(x) = 1$.

But $\lim_{x \to \infty} F(x) = b$

thus ~~this~~ $b = 1$

2> $\lim_{x \to -\infty} F(x) = 0$

But $\lim_{x \to -\infty} F(x) = a$.

Thus $a = 0$

3> Continuous and increasing.

Thus $\lim_{x \to 1^+} F(x) = \lim_{x \to 1^-} F(x)$

But $\lim_{x \to 1^+} F(x) = b = 1$

$$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} kx^2 = k.$$

$$\therefore \quad k = 1$$

Hence $a = 0$, $b = 1$, $k = 1$.

The p.d.f. is given by $f(x) = F'(x) = \begin{cases} 0 & x \le 0 \\ 2x & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$

Donot worry about boundaries ~~and~~ unless asked for.

Then. mean $= \displaystyle\int_{-\infty}^{\infty} x f(x)\, dx = \cancel{\int_{-\infty}^{0}} \dots$

$$= \cancel{\int_{-\infty}^{0} x f(x)\, dx} \to 0.$$

$$+ \int_{0}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x f(x)\, dx.$$

$$= \lim_{c\to\infty} \int_{0}^{c} x f(x)\, dx.$$

$$= \int_0^1 x f(x)\, dx + \cancel{\lim_{c\to\infty} \int_1^c x f(x)\, dx} \to 0.$$

$$= \int_0^1 x(2x)\, dx = \int_0^1 2x^2\, dx$$

$$= \left.\frac{2x^3}{3}\right|_0^1 = \frac{2}{3}.$$