Concepts: parametric representations for a circle, ellipse, parabola, line.

There are three convenient ways to write a function that relates x and y:

Explicit function: $y = f(x)$, for $a < x < b$
Implicit function: $F(x, y) = 0$ for $a < x < b$ and $c < y < d$,
Parametric function: $(x(t), y(t))$ for $a < t < b$.

All three of these can be used to create a set of ordered pairs, (x, y), which represent graphs of relations in the xy-plane.

In this course, we have seen a variety of explicit functions, for example:

Rational Function $\quad y = f(x) = \dfrac{x^2 - 1}{x + 25}$

Exponential Function $\quad y = f(x) = e^{3x}$

Polynomial Function $\quad y = f(x) = 24x^5 + 12$

and a variety of implicit functions:

Circle $\quad (x - 2)^2 + (y - 3)^2 = 9$

Hyperbola $\quad \dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

Parabola $\quad x = 4y^2$

In this section we want to take a moment to determine some important parametric representations. Having these other representations can make solving certain problems easier (for example, most of the sketches using a computer I do using parametric representations). Parametric representations are not unique, so you can come up with other ways to represent circles, ellipses, lines, etc.

Parametric Representation of a Circle

We know a circle has the implicit form $x^2 + y^2 = r^2$. The parametric form of the circle is

$$x = r\cos t$$
$$y = r\sin t, \quad 0 \le t < 2\pi$$

We can verify this represents a circle using some trig identities to eliminate the parameter t:

$$x^2 + y^2 = r^2\cos^2 t + r^2\sin^2 t = r^2(\cos^2 t + \sin^2 t) = r^2(1) = r^2$$

If we want to shift the center of the circle from the origin to (h, k), we just modify the parametrization:

$$x = h + r\cos t$$
$$y = k + r\sin t, \quad 0 \le t < 2\pi$$

Parametric Representation of an Ellipse

We know an ellipse has the implicit form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The parametric form of an ellipse is:

$$x = a\cos t$$
$$y = b\sin t, \quad 0 \le t < 2\pi$$

If we want to shift the center of the ellipse, we just modify the parametrization:

$$x = h + a\cos t$$
$$y = k + b\sin t, \quad 0 \le t < 2\pi$$

Parametric Representation of a Parabola

We know a parabola opening up has the implicit form $x^2 = 4py$. The parametric form of a parabola is:

$$x = t$$
$$y = \frac{1}{4p}t^2$$

We know a parabola opening to the right has the implicit form $y^2 = 4px$. The parametric form of a parabola is:

$$x = \frac{1}{4p}t^2$$
$$y = t$$

Shifting from $(0,0)$ to (h,k) is the same as for circles and ellipses.

Parametric Representation of a Straight Line

A straight line segment between the points (x_1, y_1) and (x_2, y_2) is given by the explicit formula:

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_2, \quad x_1 < x < x_2$$

where we have assumed $x_1 < x_2$.

Compare with the very elegant parametric representation:

$$x = (1-t)x_1 + tx_2$$
$$y = (1-t)y_1 + ty_2, \quad 0 < t < 1$$

Notice that if $t = 0$ we get $x = x_1, y = y_1$ and if $t = 1$ we get $x = x_2, y = y_2$.

If we eliminate the parameter t, we can verify this is a straight line segment with endpoints (x_1, y_1) and (x_2, y_2).

What's so Cool about Parametric Representations

Parametric representations are very useful in *dynamical systems* (something that changes with time); if we think of the parameter t as time, then $x(t)$ and $y(t)$ represent the position of the particle at time t! We can use this to construct equations that model the motion of macroscopic objects like baseballs, swaying bridges, bending poles, heat conduction through steel, even how electricity moves!

Also, they are very easy to extend to higher dimensions:

$$x = 3\cos t$$
$$y = 3\sin t$$
$$z = t$$

gives you a helix in three dimensions!

The text Section 6.3 has some examples of how you can use parametric representations to model different physical systems.