MATH1013 Calculus I

Introduction to Functions[1]

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Trigonometric Functions (Chapter 1)

[1] Based on Briggs, Cochran and Gillett: Calculus for Scientists and Engineers: Early Transcendentals, Pearson 2013

Recalling trigonometric functions

- Trigonometric ratios have been known to us as ancient as Babylonians' time because of the need of astronomical observation. The Greeks turns the subject into an exact science.
- There are many ways that these ratios appear from everyday life applications to the most advanced scientific pursues, that are generally connected with circular symmetry or with circles.
- **Earliest definitions** Given an angle A with $0 < A < \pi/2$. Then one can form a right angle triangle $\triangle ABC$ with the side a opposite to the angle A called the opposite side, the side next to the angle A called the adjacent side, and the side opposite to the right angle, the hypotenuse.
- One can certainly include the consideration that $A = 0$ or $A = \pi/2$ by accommodating a straight line section as a generalized triangle, which will be useful when we discuss trigonometry ratios for arbitrary triangles later.

Six trigonometric ratios

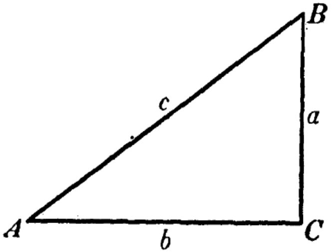


Figure: (Right angle triangle $\triangle ABC$: from Maxime Bôcher, 1914)

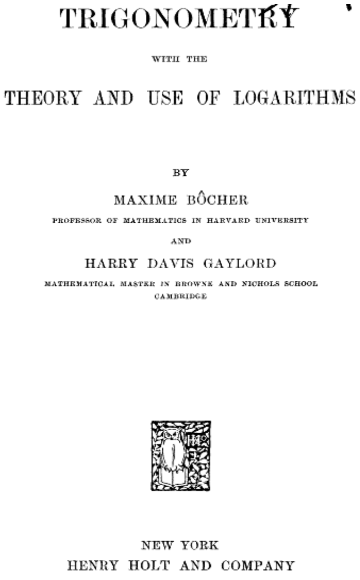
$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}, \qquad \cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}.$$

The other three are

$$\csc A = \frac{1}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \cot A = \frac{1}{\tan A}$$

Maxime Bôcher's book



TRIGONOMETRY

WITH THE

THEORY AND USE OF LOGARITHMS

BY

MAXIME BÔCHER

PROFESSOR OF MATHEMATICS IN HARVARD UNIVERSITY

AND

HARRY DAVIS GAYLORD

MATHEMATICAL MASTER IN BROWNE AND NICHOLS SCHOOL CAMBRIDGE

NEW YORK

HENRY HOLT AND COMPANY

Figure: (Bôcher's book (1914))

Maxime Bôcher's book

Maxime Bôcher (1867-1918) was an American mathematician studied in Göttingen, Germany, in late nineteenth century, then the world centre of mathematics. He later became a professor in mathematics at the Harvard university.

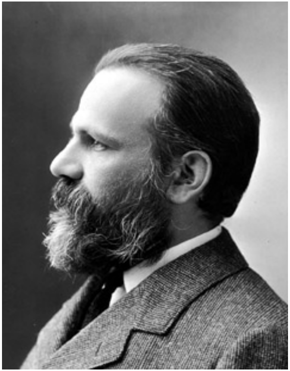


Figure: (M. Bôcher) Courtesy of Harvard University Archives

Similar triangles

CHAPTER I

FUNCTIONS OF ACUTE ANGLES. TABLES OF NATURAL FUNCTIONS. SOLUTION OF RIGHT TRIANGLES

1. Definitions. Let A (Fig. 1) be any acute angle. If from points P_1, P_2, P_3 on one side of this angle perpendiculars be dropped on the other side, right triangles AP_1R_1, AP_2R_2, AP_3R_3 are formed which are all similar to each other. Hence the ratios of corresponding sides of these triangles are equal, as, for instance:

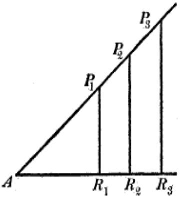


FIG. 1

(1) $$\frac{R_1P_1}{AP_1}=\frac{R_2P_2}{AP_2}=\frac{R_3P_3}{AP_3},$$

(2) $$\frac{AR_1}{AP_1}=\frac{AR_2}{AP_2}=\frac{AR_3}{AP_3},$$

(3) $$\frac{R_1P_1}{AR_1}=\frac{R_2P_2}{AR_2}=\frac{R_3P_3}{AR_3}.$$

The value of the ratios in (1) is called the Sine of A and

Figure: (Chapter One)

Trigonometric Tables

	sin	tan	sec	
5°	0.0872	0.0875	1.00	85°
10°	0.174	0.176	1.02	80°
15°	0.259	0.268	1.04	75°
20°	0.342	0.364	1.06	70°
25°	0.423	0.466	1.10	65°
30°	0.500	0.577	1.15	60°
35°	0.574	0.700	1.22	55°
40°	0.643	0.839	1.31	50°
45°	0.707	1.00	1.41	45°
50°	0.766	1.19	1.56	40°
55°	0.819	1.43	1.74	35°
60°	0.866	1.73	2.00	30°
65°	0.906	2.14	2.37	25°
70°	0.940	2.75	2.92	20°
75°	0.966	3.73	3.86	15°
80°	0.985	5.67	5.76	10°
85°	0.996	11.4	11.5	5°
	cos	ctn	csc	

Figure: (Chapter One)

It really is about circles !!

The crux of the matter of why any of the trigonometric ratios only depends on the angle and not the size of the triangle really means we are looking at a circle.

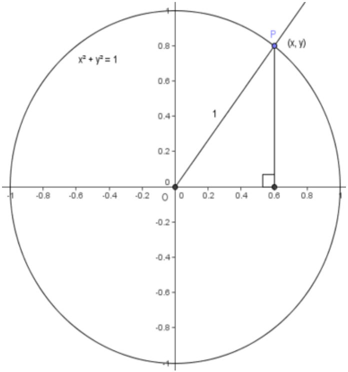


Figure: (Source: Wiki)

Trigonometric identities

- The size of the right angle triangle (circle) does not matter. One immediately obtain the famous Pythagoras theorem

$$\sin^2\theta + \cos^2\theta = 1$$

  where $0 \le \theta \le \pi/2$.
- Dividing both sides by $\sin^2\theta$ and $\cos^2\theta$ yield, respectively

$$1 + \cot^2\theta = \csc^2\theta \qquad \text{and} \qquad \tan^2\theta + 1 = \sec^2\theta.$$

- Double angle formulae:

$$\sin 2\theta = 2\cos\theta\sin\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

- Half angle formulae:

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1, \qquad \sin\theta = 1 - 2\sin^2\frac{\theta}{2}.$$

Ratios beyond $\frac{\pi}{2}$

- The question is now how to extend the angle beyond $\frac{\pi}{2}$, as the unit circle clearly indicates such possibility.

18. Functions of Angles of Any Magnitude. Let P be any point on the terminal side of the angle BAB'. From P drop a perpendicular on the initial side of the angle (or the initial side produced) meeting it in M. Then the primary trigonometric functions of the angle BAB' are defined as follows:

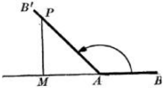


Fig. 15

$$\sin BAB' = \frac{MP}{AP},$$

$$(1) \qquad \cos BAB' = \frac{AM}{AP},$$

$$\tan BAB' = \frac{MP}{AM},$$

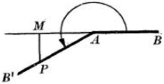


Fig. 16

where the segments AM, MP, AP are to be taken with the proper sign according to the conventions of § 17.

The secondary functions are, as in § 1, defined as the reciprocals of the primary functions:

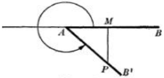


Fig. 17

$$\text{ctn } BAB' = \frac{1}{\tan BAB'},$$

$$(2) \qquad \sec BAB' = \frac{1}{\ldots}$$

Figure: (Source: Bocher, page 34)

Ratios as circular functions: $0 \leq \theta \leq 2\pi$

- Due to the propertiy of similar triangles, it is sufficient that we consider unit circle in extending to $0 \leq \theta \leq 2\pi$ with the ratios augmunted with appropriate signs from the $xy-$coordinate axes.

20. Line Values of Functions. About the vertex O of a given angle as center describe a circle of unit radius cutting the initial side in B and the terminal side in P_2 (Fig. 23). We suppose at first that the angle lies in the second quadrant. Draw M_2P_2 perpendicular to the diameter BB'. Then, by the definitions of § 18, taking account of the algebraic signs, we have

$$\sin BOP_2 = \frac{M_2P_2}{OP_2} = \frac{M_2P_2}{1} = M_2P_2,$$

$$\cos BOP_2 = \frac{OM_2}{OP_2} = \frac{OM_2}{1} = OM_2.$$

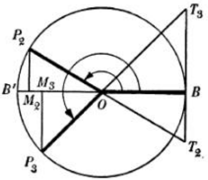


Fig. 23

Line values for the tangent and secant may be obtained by drawing at B a tangent to the circle and producing the terminal side, OP_2, backward till it meets this tangent in T_2.

We have then

Illustration of $0 \leq \theta \leq \pi$

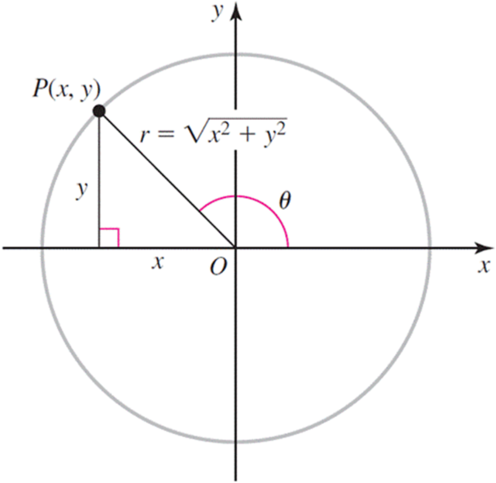


Figure: 1.59 (Publisher)

Ratios beyond 2π

- For sine and cosine functions, if θ is an angle beyond 2π, then $\theta = \phi + k\,2\pi$ for some $0 \leq \phi \leq 2\pi$. Thus one can write down their meanings from the definitions

$$\sin(\theta) = \sin(\phi + k\,2\pi) = \sin\phi, \quad \cos(\theta) = \cos(\phi + k\,2\pi) = \cos\phi$$

where $0 \leq \phi \leq 2\pi$, and k is any integer.

- The case for tangent ratio is slightly different, with the first extension to $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and then to arbitrary θ. Thus one can write down

$$\tan(\theta) = \tan(\phi + k\,\pi) = \tan\phi$$

where $\theta = \phi + k\pi$ for some k and $-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$.

Illustration of $2\pi \leq \theta \leq 4\pi$

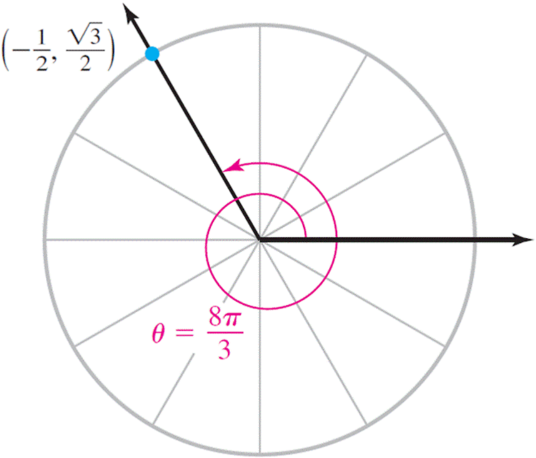


Figure: 1.61 (Publisher)

Illustration of $-4\pi \leq \theta \leq -2\pi$

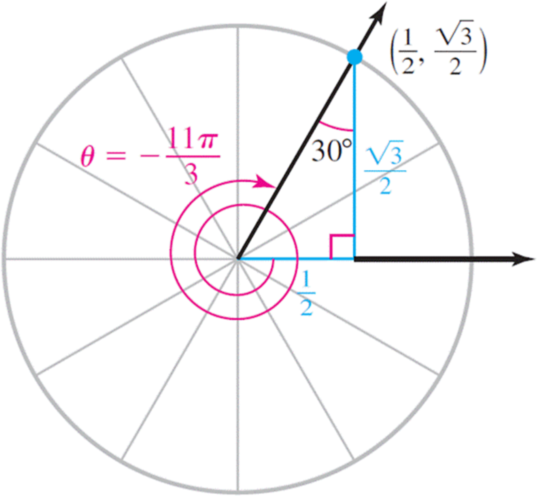
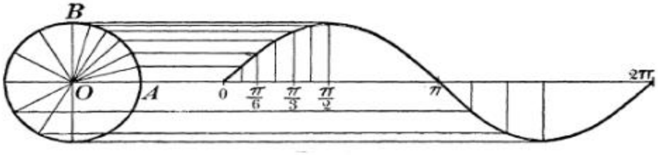
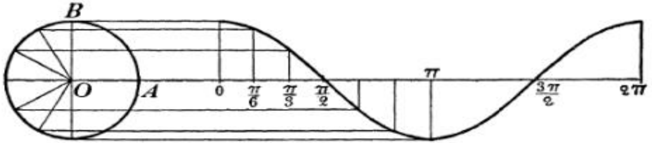


Figure: 1.62 (Publisher)

Periodic sine and cosine functions





Periodic sine/cosec functions (publisher)

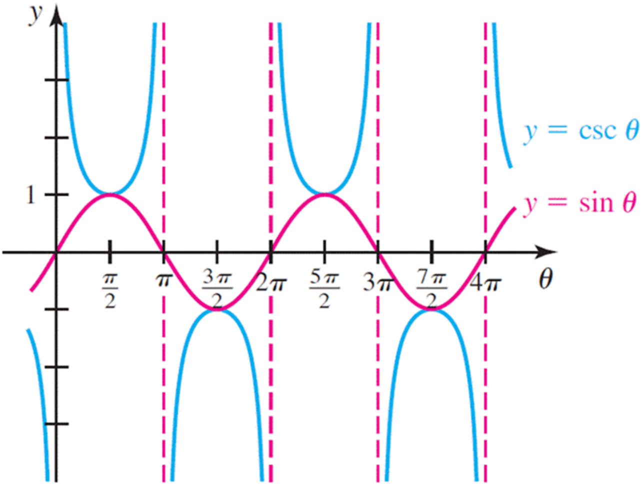


Figure: 1.63a (publisher)

Periodic cos/sec functions (publisher)

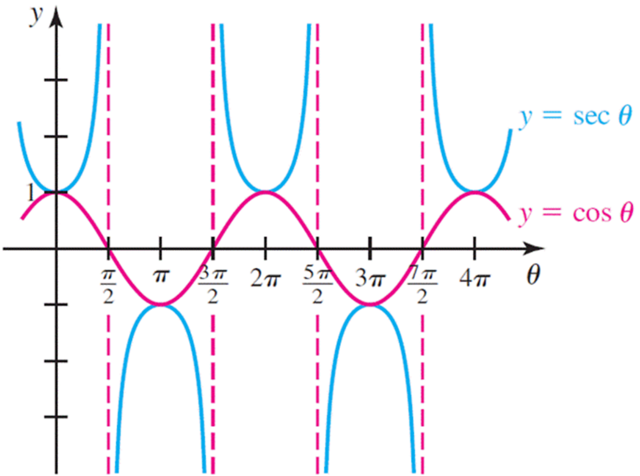


Figure: 1.63b (publisher)

Periodic tangent function

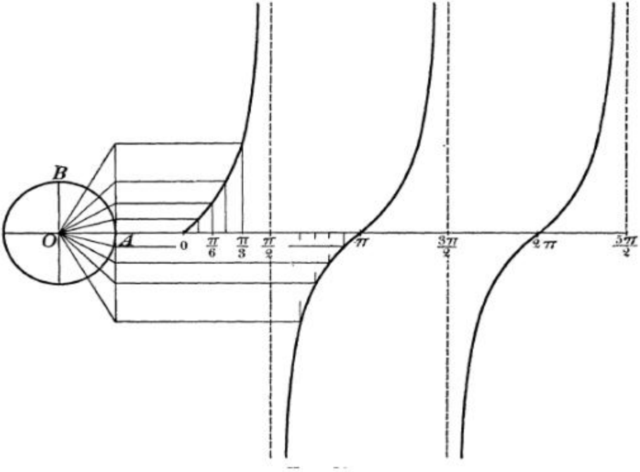


Figure: (Source: tangent, Borcher page 46)

Periodic tangent functions (publisher)

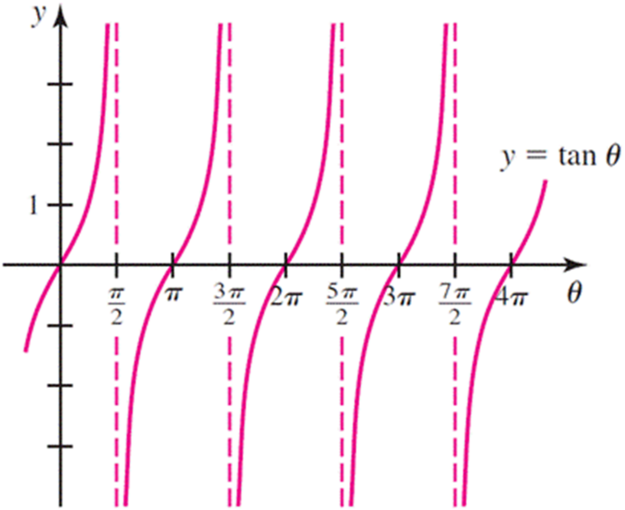


Figure: 1.64a (publisher)

Periodic cotangent functions (publisher)

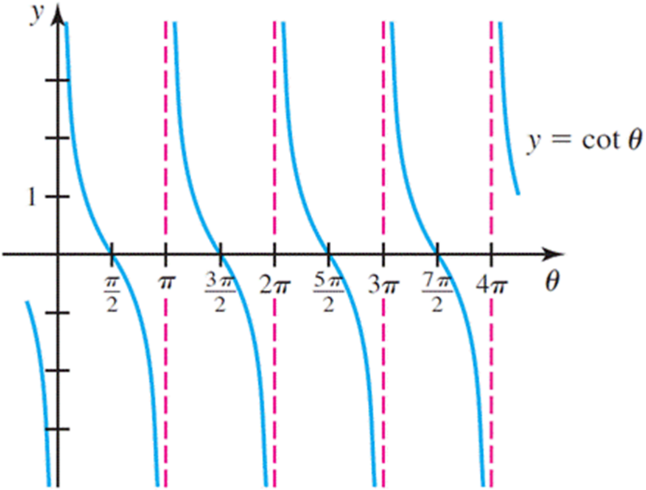


Figure: 1.64b (publisher)

Graphs of $-4\pi \leq \theta \leq -2\pi$

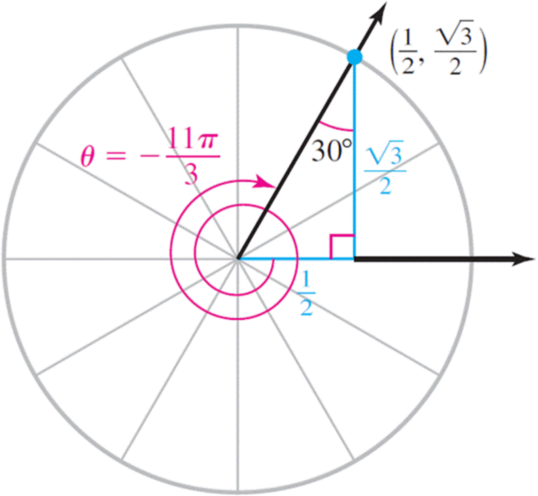


Figure: 1.62 (Publisher)

Inverse trigonometric functions

- The sine and cosine functions map the $[k\,2\pi,\ (k+1)\,2\pi]$ onto the range $[-1,\ 1]$ for each integer k. So it is

  $$\textbf{many} \longrightarrow \textbf{one}$$

  so an inverse would be possible only if we suitably restrict the domain of either sine and cosine functions. We note that even the image of $[0,\ 2\pi]$ "covers" the $[-1,\ 1]$ more than once.
- In fact, for the sine function, only the subset $[\frac{\pi}{2},\ \frac{3\pi}{2}]$ of $[0,\ 2\pi]$ would be mapped onto $[-1,\ 1]$ exactly once. That is the sine function is one-one on $[\frac{\pi}{2},\ \frac{3\pi}{2}]$.
- However, it's more convenient to define the inverse $\sin^{-1} x$ on $[-\frac{\pi}{2},\ \frac{\pi}{2}]$.

  $$\sin^{-1} x : \quad [-1,\ 1] \longrightarrow \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right].$$

- $$\cos^{-1} x : \quad [-1,\ 1] \longrightarrow [0,\ \pi].$$

Inverse sine (publisher)

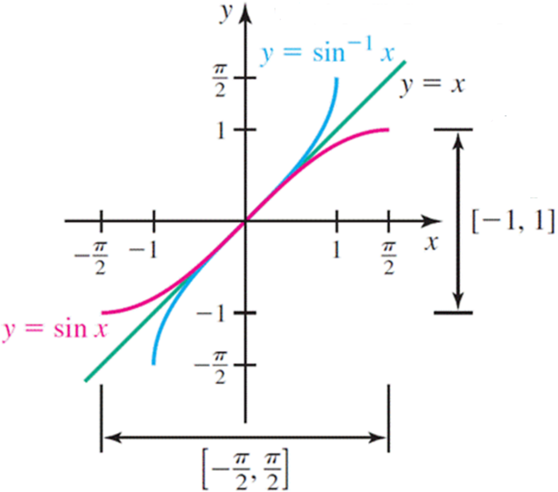


Figure: 1.71a (publisher)

Inverse cosine (publisher)

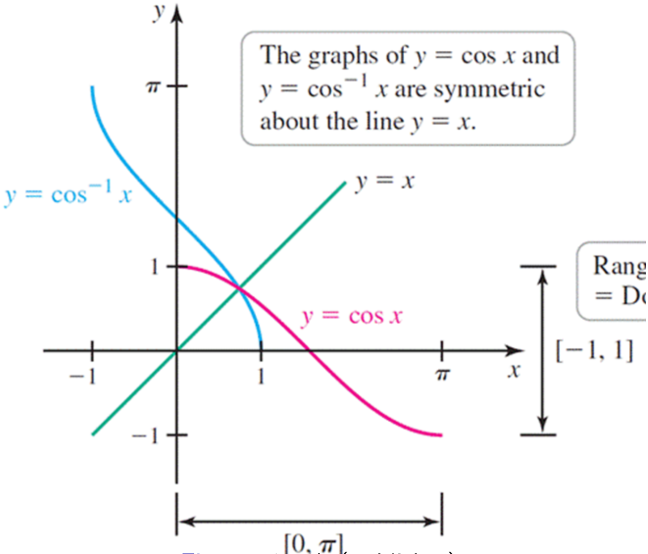


Figure: 1.71b (publisher)

Inverse trigonometric functions II

- The tangent has

$$\tan^{-1} x : \quad (-\infty, \infty) \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

  or the whole real axis.
- (p. 44) Find (a) $\cos^{-1}(-\sqrt{3}/2)$, (b) $\cos^{-1}(\cos 3\pi)$, (c) $\sin(\sin^{-1} 1/2)$.
- (p. 45) Given $\theta = \sin^{-1} 2/5$. Find $\cos\theta$ and $\tan\theta$.
- (p. 45) Find alternative form of $\cot(\cos^{-1}(x/4))$ in terms of x.
- (p. 48, Q. 58) Find $\cos(\sin^{-1}(x/3))$.
- (p. 45) Why $\sin^{-1} x + \cos^{-1} x = \pi/2$?

Inverse tangent (publisher)

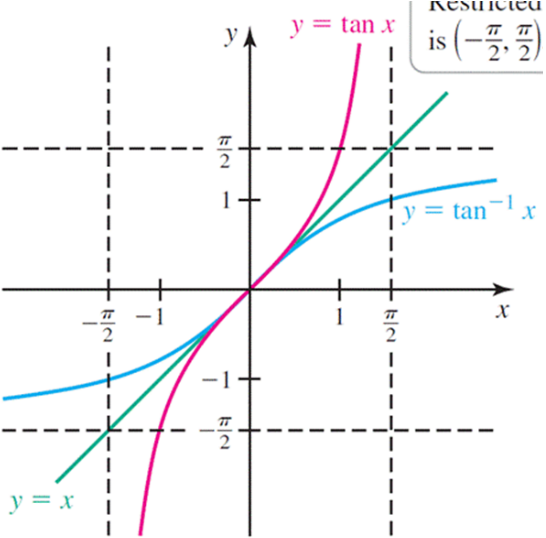


Figure: 1.77 (publisher)

Inverse cotangent (publisher)

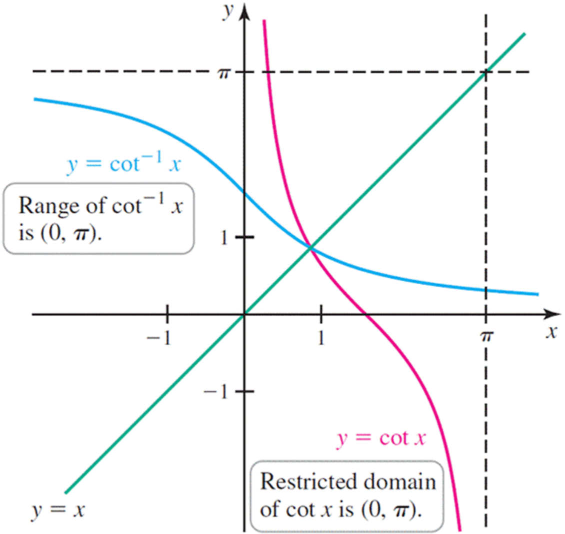


Figure: 1.78 (publisher)

Inverse secant (publisher)

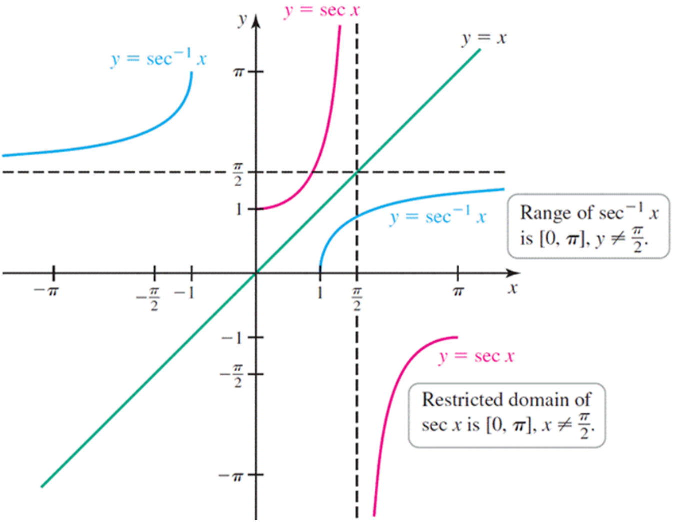


Figure: 1.79 (publisher)

Inverse cosec (publisher)

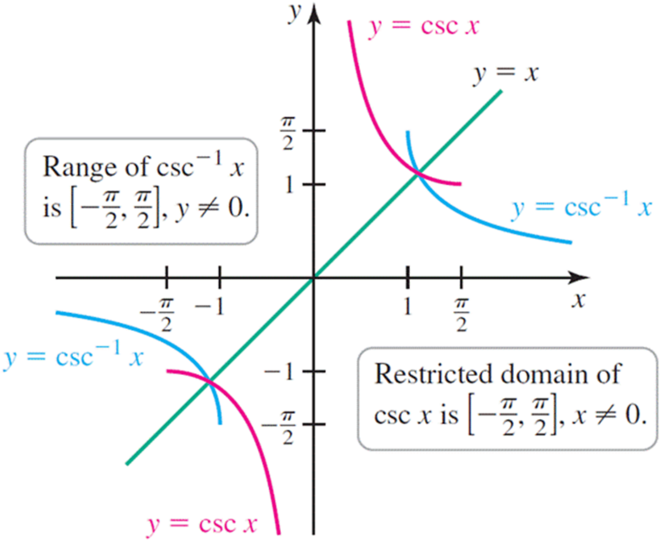


Figure: 1.80 (publisher)