Uniqueness and Non-Uniqueness of Measures of Maximal Entropy

Based on the lecture notes by R. Pavlov

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To be precise, we will show that for *uniform Gibbs measures*.

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They are uniform Gibbs measures and we will show that for $M >> 1$ they are indeed different.

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To show this we will prove that $\mathbb{P}(x_{(0,0)} > 0 \mid x|_{\partial\{-n,\dots,n\}^2} \equiv -M) < \frac{1}{2} - \epsilon$.

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Then obviously $\mathbb{P}(x_{(0,0)} > 0 \mid x|_{\partial\{-n,\dots,n\}^2} \equiv M) > \frac{1}{2} - \epsilon$
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For any x such that $x_{(0,0)} > 0$ and $x|_{\partial\{-n,\dots,n\}^2} \equiv -M$ there is maximal path-connected component of $\{-n, \dots, n\}^2$ on which x is positive.

Call it P_x and let $B_x := \partial P_x$.

By the definition of $\mathcal{I}_M$ we have $x|_{B_x} \equiv 1$ and $x|_{\partial(\mathbb{Z}^2 \setminus P_x)} \equiv -1$.

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List all possible boundaries $B_1, \ldots, B_k$ and let E_i be set of those $x \in \mathcal{A}^{\{-n,\ldots,n\}^2}$ for which $B_x = B_i$.

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For $x \in E_i$ we can form an admissible element x' such that $x|_{\mathbb{Z}^2 \setminus P_x} = x'|_{\mathbb{Z}^2 \setminus P_x}$, $x'|_{P_x} \equiv -x|_{P_x}$ and $x'|_{B_x} < 0$ but arbitrary otherwise.

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It follows that $\mathbb{P}(E_i \mid x|_{\partial\{-n,\ldots,n\}^2} \equiv -M) < M^{-|B_i|}$ and further that

$$\mathbb{P}(x_{(0,0)} > 0 \mid x|_{\partial\{-n,\ldots,n\}^2}) = \ldots$$

$$\ldots = \sum_{i=1}^{k} \mathbb{P}(E_i \mid x|_{\partial\{-n,\ldots,n\}^2} \equiv -M) \leq \sum_{i=1}^{k} M^{-|B_i|}.$$

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But for fixed j we have less than $j4^j$ possible B_i with $|B_i| = j$ and therefore

$$\mathbb{P}(x_{(0,0)} > 0 \mid x|_{\partial\{-n,\dots,n\}^2}) \leq \sum_{i=1}^{k} M^{-|B_i|} < \sum_{j=1}^{\infty} j4^j M^{-j} = \dots$$

$$\dots = (\sum_{j=1}^{\infty} jx^j)|_{x=\frac{4}{M}} = \frac{4M}{M^2 - 8M + 16} \to 0 \text{ for } M \to \infty.$$

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Let μ and ν be uniform Gibbs measures on an STF X such that $(\mu \times \nu)$-a.e. pair $(x, y) \in X \times X$ patterns x and y agree somwhere on every infinite path.
Then $\mu = \nu$.

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For $(x, y) \in [w] \times (X \setminus [w])$ let $C_{(x,y)}$ be the maximal path-connected set intersecting w on which x and y disagree.

By our assumption for $(\mu \times \nu)$-a.e. pair $(x, y) \in X \times X$ $C_{(x,y)}$ is bounded and so there is a path surrounding w on which x and y agree.

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Define $\Phi : X \times X \to X \times X$ by $\big(\Phi(x), \Phi(y)\big)|_{C_{(x,y)}} = \big(y, x\big)|_{C_{(x,y)}}$.

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We also have $\Phi\big([w] \times (X \setminus [w])\big) = (X \setminus [w]) \times [w]$, implying

$(\mu \times \nu)\big([w] \times (X \setminus [w])\big) = (\mu \times \nu)\big((X \setminus [w]) \times [w]\big)$.

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Then $\mu = \nu$.

Therefore we get

$$\mu([w]) = (\mu\times\nu)\big([w]\times X\big) = (\mu\times\nu)\big([w]\times(X\setminus[w])\big)+(\mu\times\nu)\big([w]\times[w]\big) = \dots$$

$$\dots = (\mu\times\nu)\big((X\setminus[w])\times[w]\big)+(\mu\times\nu)\big([w]\times[w]\big) = (\mu\times\nu)\big(X\times[w]\big) = \nu([w]).$$

Since cylinders generate the topology, that concludes the proof.

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It follows that for $S \subset \mathbb{Z}^2$ $(\mu \times \nu)\big(x|_S \neq y|_S\big) \leq \mu_{\frac{1}{2}}(x|_S = 0)$.

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Define $\Theta(p) := \mu_p(|C_p| = \infty)$ and $p_c := \sup\{p \in [0,1] : \Theta(p) = 0\}$.

Harris and Hammersley showed that $p_c > \frac{1}{2}$ and this is precisely what we need to finish the proof.

References

R. Pavlov. *Entropy and mixing for $\mathbb{Z}^d$ STFs*. First School on Dynamical Systems and Computation. CMM, Santiago, Chile