Lecture 4: An introduction to hom-shifts

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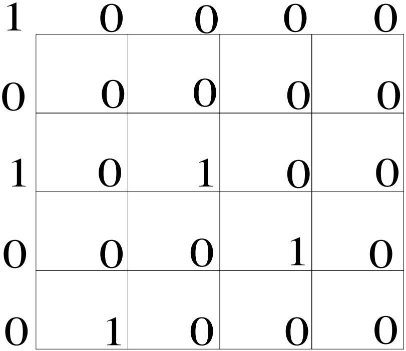
Recall: Shifts of finite type

An SFT is a space of configurations $X \subset \mathbb{A}^{\mathbb{Z}^d}$ which can be obtained by forbidding a finite set of patterns.

A nearest neighour SFT is an SFT which can be obtained by forbidding patterns on edges of $\mathbb{Z}^d$.

Our beloved examples: Hard core shift

Here $\mathbb{A} = \{0, 1\}$ and adjacent symbols can't both be one.



Our beloved examples: k-colorings

Here $\mathbb{A} = \{1, 2, 3, \ldots, k\}$ and adjacent symbols can't both be the same.

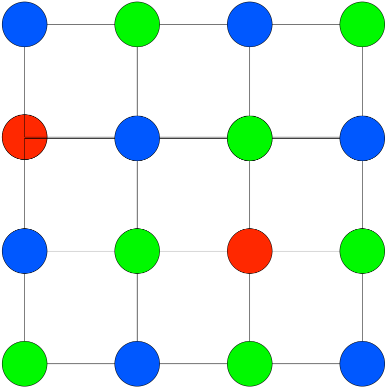


Figure : A 3-colouring

Our beloved examples: Iceberg shift

Here $\mathbb{A} := \{-m, -m+1, \ldots, -1, 1, \ldots, m\}^{\mathbb{Z}}$ and adjacent symbols can't have the opposite signs unless they are 1 and -1.

-2	-3	-1	-1	-2	-1	1	1	-1	-2	-2
-1	-1	-2	-2	-1	1	4	1	-1	-2	-2
-5	-2	-6	-1	1	5	3	1	-1	-2	-2
-1	-2	-1	-1	1	3	6	1	-1	-2	-2
1	-1	1	1	2	5	1	-1	-2	-2	-2
2	1	2	2	3	1	-1	-2	-2	-2	-2
5	6	7	3	4	1	-1	-2	-2	-2	-2
1	1	1	4	5	1	-1	-2	-2	-2	-2
-1	-1	-1	1	1	-1	-2	-2	-2	-2	-2
-2	-2	-2	-1	-1	-2	-2	-2	-2	-2	-2
-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2

Figure : Iceberg Shift

Measures of maximal entropy

1. The function μ to h_μ is upper semi-continuous and achieves a maximum. The measures which achieve them are called measures of maximal entropy.
2. If X is a nearest neighbour SFT then every measure of maximal entropy is a uniform Gibbs measure, that is, for all finite sets $B \subset \mathbb{Z}^d$ and $y \in supp(\mu)$ and $x \in X$,
$$\mu([x]_B \ : \ [y]_{\mathbb{Z}^d \setminus B}) = \mu([x]_B \ : \ [y]_{\partial B})$$
and is uniform on $x \in X$ such that $x|_{\mathbb{Z}^d \setminus B} = y|_{\mathbb{Z}^d \setminus B}$.
3. A measure on the hard core shift and the iceberg shift is a measure of maximal entropy if and only if it is uniform Gibbs.

Unique measures of maximal entropy

1. The hard core shift has a unique measure of maximal entropy when $d = 2$. (van den Berg and Steif 1994).
2. Domino tilings have a unique measure of maximal entropy for $d = 2$ (Cohn, Kenyon and Propp 2001—but was probably known before)
3. The space of proper 3-colourings has a unique measure of maximal entropy (Sheffield 2006)

Question

Prove that the measure of maximal entropy is unique for all dimensions for the hard-core shift.

Multiple ergodic measures of maximal entropy

1. The iceberg example has exactly 2 measures of maximal entropy for large enough M.
2. Peled and Spinka (2018) proved that for all k, for large enough d, there are exactly $\binom{k}{\lfloor k/2 \rfloor}$ number of "periodic" measures of maximal entropy.

Where does $\binom{k}{\lfloor k/2 \rfloor}$ come from?

Let us first take a step back.

In general the set of measures of maximal entropy can be a very large set.

A weird example

Take $X \subset \{1,2\}^{\mathbb{Z}^2}$ where $\begin{smallmatrix}2\\1\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix}$ are disallowed.

1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2
1	1	2	1	2	2	1	1	2

Figure : The columns are constant

Clearly fixing up the pattern on any horizontal line fixes up everything. Thus $|\mathcal{L}(X, \{1, 2, 3, \ldots, n\}^2)| \leq \{1, 2\}^n$.

Thus $h_{\text{top}}(X) := 0$ and so every measure on X is an mme.

Each ergodic measure on $\{1, 2\}^{\mathbb{Z}}$ determines an ergodic measure on X (by making it constant on columns). Each such measure gives a distinct measure of maximal entropy.

There are uncountably such measures (why?).

We want to avoid such examples.

Hom-Shifts

Graph homomorphisms and hom-shifts

A graph homomorphism is a map between graphs which preserves adjacencies. Given a graph H, a hom-shift associated with H is the space

$$X_H^d = \{\text{graph homomorphisms from } \mathbb{Z}^d \text{ to } H\}.$$

There are nearest neighbour shifts of finite type which are invariant under permutations of coordinates and reflections.

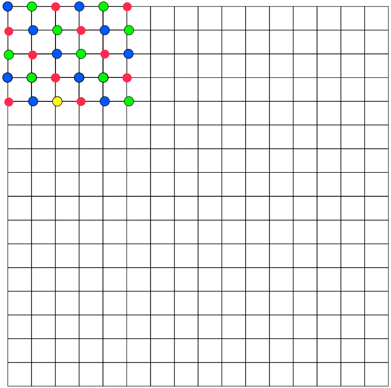
Some features of hom-shifts

There are numerous features of hom-shifts which make them more tractable. If symbol *a* can sit next to *b* in some direction then it can sit next to *a* in all directions.

a	b	a	b	a	b
b	a	b	a	b	a
a	b	a	b	a	b
b	a	b	a	b	a
a	b	a	b	a	b

Some features of hom-shifts

Thus given any pattern from a hom-shift, its reflection is also a valid pattern. By multiple reflections of patterns on a rectangular box we get periodic configurations. Thus periodic configurations are dense.



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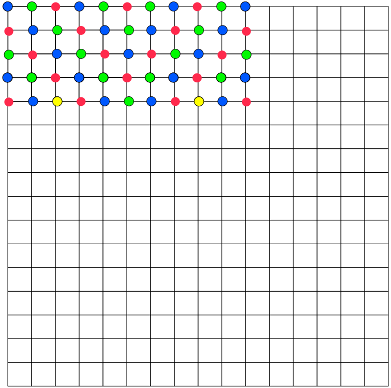

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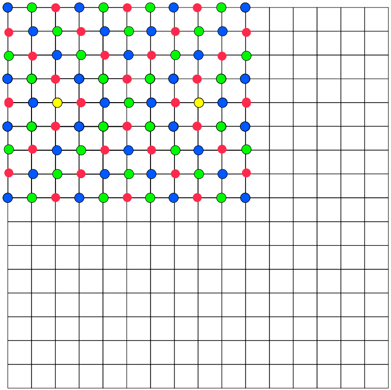

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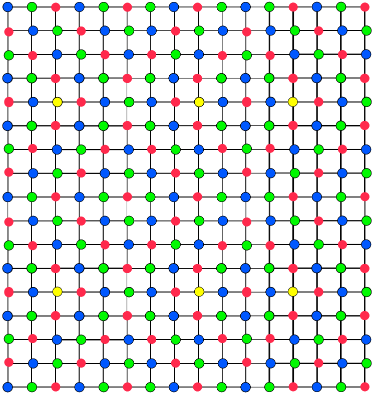


Some examples: Full shifts

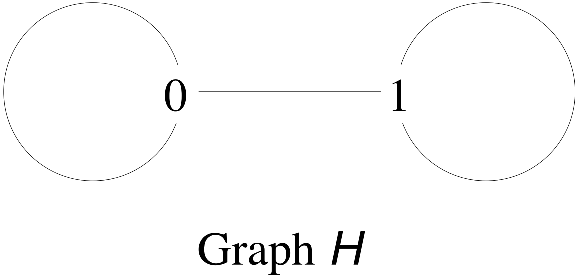


Figure : No Constraints

Measures of maximal entropy for the full shift

Measures of maximal entropy of nearest neighbour SFTs are uniform Gibbs measures.

Thus for the full shift $\{0,1\}^{\mathbb{Z}^d}$, and $x, y \in \{0,1\}^{\mathbb{Z}^d}$ and a finite set $A \subset \mathbb{Z}^d$ we have

$$\mu([x]_A \mid [y]_{\mathbb{Z}^d \setminus A}) = \frac{1}{2^{|A|}}.$$

Thus

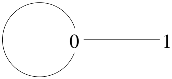
$$\mu([x]_A) = \frac{1}{2^{|A|}}.$$

This is the same as choosing either 0 or 1 at each site in $\mathbb{Z}^d$ independently and with equal probability.

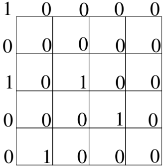
The measure of maximal entropy is unique.

Hom-shift: The hard-core shift

The symbols are 0 and 1. Adjacent symbols in this shift cannot be both 1. This is the space X_H^d where H is given by the graph below.



Graph $\mathcal{H}$



Hom-shift: The hard-core shift

Van den Berg and Steif showed in 1994 that there is a unique measure of maximal entropy. Is it unique in all dimensions?

Hom-shifts: Proper q-colorings

The symbols are $1, 2, 3, \ldots, q$. Adjacent symbols in this shift are distinct. This is the space $X^d_{K_q}$ where K_q is the complete graph on q vertices.

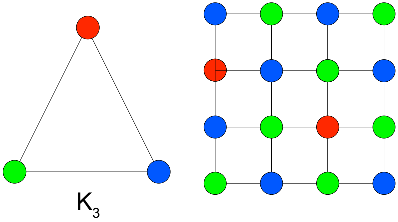


Figure : Proper 3-colorings

Measures of maximal entropy for proper q-colourings

X_3^2 has a unique mme (Sheffield 2006).

$X_{K_q}^d$ has a unique measure of maximal entropy for $q > 4d + 2$ (Dobrushin 1968).

X_q^d has a unique measure of maximal entropy for $q > 3.6d$. (Gamarnik, Katz, and Misra 2015)

Question

Prove that there is q_d such that $q_d/d \to 1$ and for $q > q_d$, X_q^d has a unique measure of maximal entropy. What about $q \geq d + 2$?

In 2018, Peled and Spinka proved that there are $\binom{q}{\lfloor \frac{q}{2} \rfloor}$ "periodic" ergodic measures of maximal entropy when q is fixed and d is large. Why $\binom{q}{\lfloor \frac{q}{2} \rfloor}$?

Hom-shifts: Iceberg shift

The symbols are $-M, -M+1, \ldots, -1, 1, \ldots, M$ and a, b, c, d, e. The only restriction is that the positives cannot sit next to the negatives.

-2	-3	-1	-1	-2	-1	1	1	-1	-2	-2
-1	-1	-2	-2	-1	1	4	1	-1	-2	-2
-5	-2	-6	-1	1	5	3	1	-1	-2	-2
-1	-2	-1	-1	1	3	6	1	-1	-2	-2
1	-1	1	1	2	5	1	-1	-2	-2	-2
2	1	2	2	3	1	-1	-2	-2	-2	-2
5	6	7	3	4	1	-1	-2	-2	-2	-2
1	1	1	4	5	1	-1	-2	-2	-2	-2
-1	-1	-1	1	1	-1	-2	-2	-2	-2	-2
-2	-2	-2	-1	-1	-2	-2	-2	-2	-2	-2
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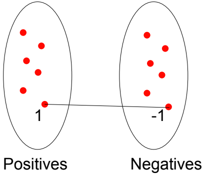

Every positive number is connected with every other positive number and every negative number is connected with every other negative number. -1 is connected to 1

Figure : Iceberg shift

Measures of maximal entropy for the iceberg shift

There are exactly two ergodic measures of maximal entropy.

Complete bipartite graph

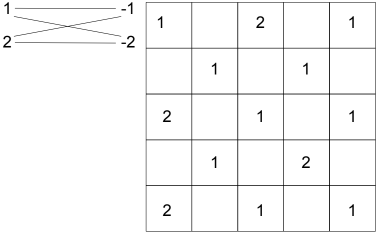

Figure : Complete bipartite graphs

What are the measures of maximal entropy like? Toss a coin and with equal probability uniformly colour the even vertices of 1 and 2 and the odd vertices with -1 and -2 or the reverse.

Complete bipartite graph

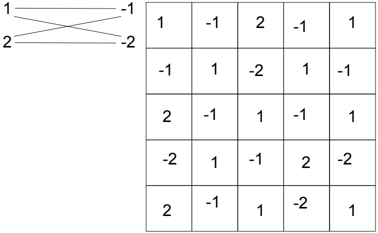

Figure : Complete bipartite graphs

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Multiple complete bipartite graph

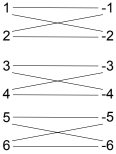


Figure : Complete bipartite graphs

What are the ergodic measures of maximal entropy like?

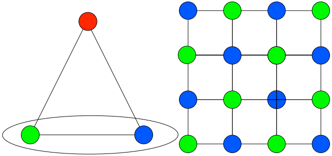
Toss a coin and with equal probability uniformly colour the even vertices of 1 and 2 and the odd vertices with -1 and -2 or the reverse. We can replace $1, 2, -1, -2$ by $3, 4, -3, -4$ etc. If there are k such copies then there are k ergodic measures of measures of maximal entropy.

Can we say something in more generality?

How to build a graph homomorphism?

Before we discuss more about this question, let us do some elementary exercises. Suppose that we are given a graph H. How can one quickly build a graph homomorphism from $\mathbb{Z}^d$ to H?

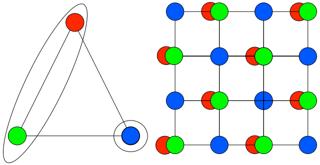
Choose two adjacent vertices of H and alternate between them.



But this has entropy zero.

A little more randomness

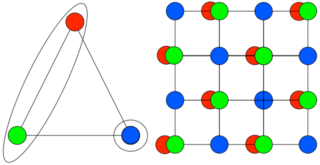
A phase of a graph H is an unordered pair of subsets (A, B) such that every vertex in A is adjacent to every vertex in B. Thus now we can build graph homomorphisms by alternating between elements of A and elements of B.



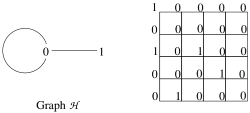
A phase is maximal if it maximises $|A||B|$. We believe that the number of maximal phases essentially dictates the number of ergodic measures of maximal entropy.

Maximal phases and the number of ergodic measures of maximal entropy

Every phase (A, B) gives rise to an ergodic measure: Place with uniform probability elements of A and B on different partite classes of $\mathbb{Z}^d$. This measure has entropy $\frac{1}{2}\log(|A||B|)$.



Maximal phases: Hard Core model



There is a unique maximal phase: $\{0, 1\}, \{0\}$.

For $d = 2$, (van den Berg and Steif 1994) it is known that the hard square model has a unique measure of maximal entropy.

In higher dimensions there are two "periodic" mmes: In one such measure $\{0, 1\}$ concentrates on the even vertices and $\{0\}$ on the odd vertices and in the other the opposite happens.

Some examples: Iceberg shift (Burton and Steif, 1994)

For M large enough the maximal phases are

$$(\{-M, \dots, -1\}, \{-M, \dots, -1\})$$

and

$$(\{1, \dots, M\}, \{1, \dots, M\}).$$

For M large enough there are exactly two measures of maximal entropy.

Some examples: Proper colourings

The symbols are $1, 2, 3, \ldots, q$. Adjacent symbols in this shift are distinct. This is the space $X^d_{K_q}$ where K_q is the complete graph on q vertices.

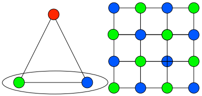


Figure : Proper 3-colorings

Maximal phases are partitions of $1, 2, \ldots q$ into cardinalities $\lfloor q/2 \rfloor$ and $\lceil q/2 \rceil$.
Peled and Spinka (2018) show that there are $\binom{q}{\lfloor 1/2 \rfloor}$ "periodic" measures of maximal entropy with precisely this split up among the odd and the even vertices.

Maximal phases and the number of ergodic measures of maximal entropy

Let $\mathbb{Z}^\infty$ (set of integer sequences which converge to zero) denote the direct limits of $\mathbb{Z}^d$ as $d \to \infty$ and let X_H^∞ denote the corresponding hom-shift.

Theorem (Pavlov, Meyerovitch 2014)

The only ergodic measures of maximal entropy of X_H^∞ invariant to a finite change of coordinates are those which arise from maximal phases.

Informal Idea

Theorem (Pavlov, Meyerovitch 2014)

The only ergodic measures of maximal entropy of X_H^∞ invariant to a finite change of coordinates are those which arise from maximal phases.

The main idea is that if we choose a configuration $x \in X_H^\infty$ according to the measure of maximal entropy then $x_{\vec{0}}$ has to be adjacent to $x_{\vec{e}_1}, x_{\vec{e}_2}, \ldots$ in H.

Note that $x_{\vec{e}_1}, x_{\vec{e}_2}, \ldots$ has the same distribution as $x_{\vec{e}_{p(1)}}, x_{\vec{e}_{p(2)}}, \ldots$ where $p : \mathbb{N} \to \mathbb{N}$ is any finite permutation. By the de Finetti's theorem $x_{\vec{e}_1}, x_{\vec{e}_2}, \ldots$ must be a mixture of iid processes.

Informally

By the law of large numbers, $x_{\vec{e}_1}, x_{\vec{e}_2}, \ldots$ take values in a set A and $x_{\vec{0}}$ takes values in a set B such that every vertex of A is adjacent to every vertex of B.

If the vertices in A and vertices in B are chosen uniformly and independently the entropy is $1/2 \log(|A||B|)$ it is clear that we have to maximise $|A||B|$. This is how maximal phases arise.

However this in infinite dimensions. This is much more complicated in finite dimensions. Is there a suitable version of de Finetti's theorem which we can apply in this situation?

We have no idea.

Conjecture

But then I feel strongly enough to conjecture.

Question

Prove that the number of ergodic measures of maximal entropy for hom-shifts are finite.

This is a far-reaching generalisation of many results mentioned in this talk. The saving grace is that I only want to prove that they are finite and not give exact bounds.

But what about other properties of mmes?

There are several other properties of mmes that interest us.

1. Are they strong mixing? Are they isomorphic to an iid process? Do they exhibit some periodicity?
2. Do they have full support?
3. Are there measures which are local maximas but not global maximas?
4. Can we compute their entropy?

Supports of mmes

We will focus for a bit on the support of mmes.

Recall, that given a probability measure μ on X, the support of the measure μ is given by

$$supp(\mu) := X \setminus \cup[a]_B$$

where the union is over cylinder sets $[a]_B$ for which $\mu([a]_B) = 0$.

It is the smallest closed set with probability one.

Why care about supports of mmes?

A shift space X is called entropy minimal if and only if for all $Y \subsetneq X$, $h_{\text{top}}(Y) < h_{\text{top}}(X)$.

If a shift space is entropy minimal then each word is indespensible.

Theorem

A shift space X is entropy minimal if and only if the support of all the mmes is X.

Why?

Entropy minimality and mmes

Theorem

A shift space X is entropy minimal if and only if the support of all the mmes is X.

Suppose X is not entropy minimal. Then there exist $Y \subsetneq X$ with the same entropy. The mme for Y is also an mme for X and is supported strictly inside X

Suppose that the support of an mme is $Y \subsetneq X$. But then $h_{\text{top}}(Y) = h_{\text{top}}(X)$.

Conjecture

Question

Prove that if H is a connected graph then the hom-shift X_H^d is entropy minimal.

There are some partial results in a paper of mine under some assumptions on H.

Hard-core shift

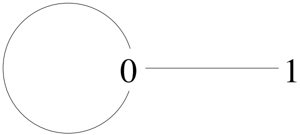
Recall, that for the hard-core shift, a measure is an mme if and only if it is a shift-invariant uniform Gibbs measure.

Now recall the following very important property of the hard-core shift.

Definition

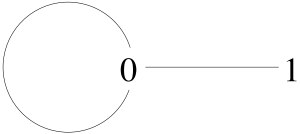
A shift space X is strongly irreducible (SI) if there is an integer N such that any two patterns $a \in \mathcal{L}(X, A)$ and $b \in \mathcal{L}(X, B)$ separated by distance greater than N there exists $x \in X$ such that $x|_A = a$ and $x|_B = b$.

Hard-core shift is strongly irreducible(SI)



0	0	1	0	1	0	1	0	1	0	a
1									1	
0		0	1	0	1	0	1		0	
1		1	0	1	0	1	0		1	
0		0	1	0	1	0	1		0	
1		1	0	1	0	1	0		1	b
0		0	1	0	1	0	1		0	
0		1	0	1	0	1	0		1	
1									0	
0	1	0	1	0	1	0	1	0	1	

Hard-core shift is strongly irreducible(SI)



0	0	1	0	1	0	1	0	1	0	a
1	0	0	0	0	0	0	0	0	1	
0	0	0	1	0	1	0	1	0	0	
1	0	1	0	1	0	1	0	0	1	
0	0	0	1	0	1	0	1	0	0	
1	0	1	0	1	0	1	0	0	1	b
0	0	0	1	0	1	0	1	0	0	
0	0	0	0	1	0	0	0	0	1	
1	0	0	0	0	0	0	0	0	0	
0	1	0	1	0	1	0	1	0	1	

Boxes in $\mathbb{Z}^d$

$$B_n := \{-n, -n+1, \ldots, n\}^d.$$

SI shifts (and hence the hard-core shift) are entropy minimal

Theorem

A shift space X is entropy minimal if and only if the support of all its mmes is X.

Let X be an SI nearest neighbour SFT and N be a distance at which we are allowed to glue patterns.

Let μ be an MME. Take some $[b]_{B_{n+N+1}}$ such that

$$\mu([b]_{B_{n+N+1}}) > 0.$$

μ is a uniform Gibbs measure. Now for all $c \in \mathcal{L}(X, B_n)$ there exists $x_c \in X$ such that $x_c|_{B_n} = c$ and $x_c|_{\partial B_{n+N}} = b|_{\partial B_{n+N}}$.

SI shifts (and hence the hard-core shift) are entropy minimal

Theorem

A shift space X is entropy minimal if and only if the support of all the mmes is X.

Since μ is a uniform Gibbs measure we have that $\mu([x_c]_{B_n} \mid [b]_{\partial B_{n+N}}) > 0$. Thus

$$\mu([c]_{B_n}) = \mu([x_c]_{B_n}) > 0.$$

μ is fully supported and X is entropy minimal. The hard-core shift is SI. Thus it is entropy minimal.

Entropy minimality of SI shifts

In fact the SFT assumption is not required (Schraudner 2010).

SI subshifts in proper 3-colourings

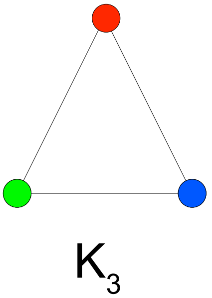
However even if H is a connected graph the hom-shift X_H^d need not be SI (in fact even contain a subshift which is SI).

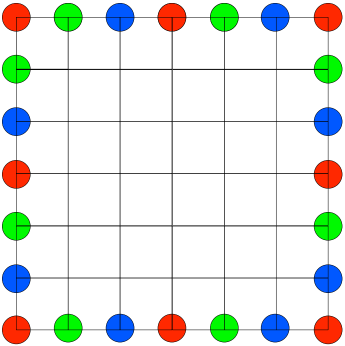
Theorem

(Chandgotia and Meyerovitch, 2019) The space of proper 3-colourings does not contain a subshift which is SI.

Proper 3-colourings are not SI

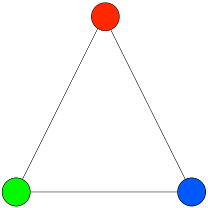
Patterns on the boundary can force the pattern inside.



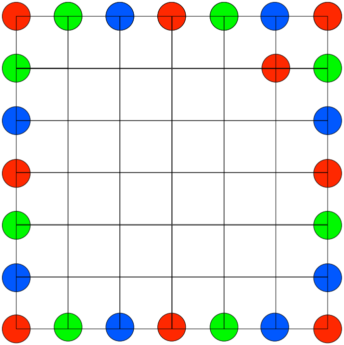


Proper 3-colourings are not SI

Patterns on the boundary can force the pattern inside.



K_3



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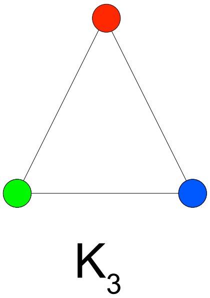

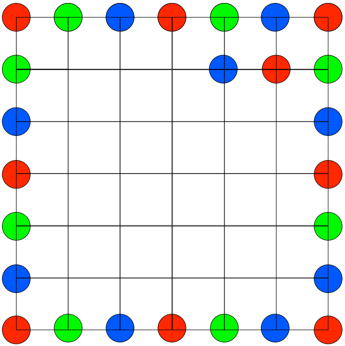

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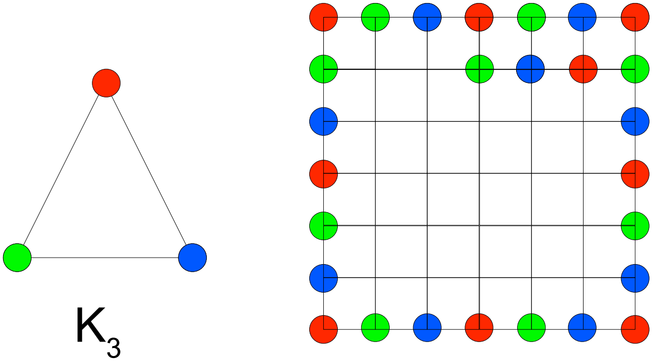

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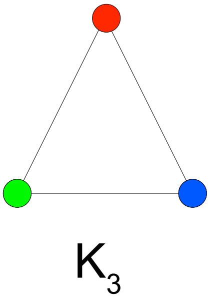

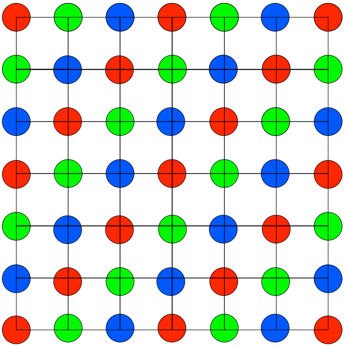

There are fully supported mmes for hom-shifts

Theorem

Let H be a connected graph. Then there exists a mme μ which is fully supported.

I proved this along with Ron Peled recently. To prove entropy minimality we need that the support of every mme is full. This we don't know how to do.

The main idea that we use is reflection positivity.

Computation of topological entropy

Let us switch gears a little bit. We had discussed earlier that the entropy of SFTs is right recursively enumerable, that is, there is an algorithm which can approximate the entropy for above.

Theorem (Friedland, 1997)

The entropy of hom-shifts is computable, there exists an algorithm which can approximate the entropy both from above and from below.

Friedland did not call them hom-shifts and his results were more general.

We can now give a short proof using reflection positivity.

Universality of hom-shifts

Let (X, μ, T) be a probability preserving action of $\mathbb{Z}^d$.

It is free if $\mu(\{x \ : \ T^{\vec{i}}(x) = x\} := 0$ for all $\vec{i} \in \mathbb{Z}^d \setminus \{\vec{0}\}$.

Definition

A shift space (X, σ) is called universal if for all free (Y, μ, S) with strictly smaller entropy there exists an equivariant injection from Y to X.

Krieger (1971) in his celebrated result showed that full shifts are universal.

Ayșe Șahin and Robinson (2000) showed that SI SFTs are universal (when they have periodic points.) Do there always exist periodic points for SI SFTs (open for $d > 2$)?

Universality of hom-shifts

Definition

A shift space (X, σ) is called universal if for all free (Y, μ, S) with strictly smaller entropy there exists an equivariant injection from Y to X.

Theorem (Chandgotia and Meyerovitch 2019)

Let H be a connected graph. Hom-shifts X_H^d are universal if and only if H is not bipartite.

The proof is complicated. However the main combinatorial estimate follows from reflection positivity.

One technique and three results: Reflection Positivity

Theorem (Friedland, 1997— we give a new proof)

The entropy of hom-shifts is computable, there exists an algorithm which can approximate the entropy from both from above and from below.

Theorem (Chandgotia and Peled)

Let H be a connected graph. There exists a mme μ for X_H^d which is fully supported.

Theorem (Chandgotia and Meyerovitch 2019)

Let H be a connected graph. Hom-shifts X_H^d is universal if and only if H is not bipartite.

What do we need to prove: Computability of entropy

Recall that

$$Loc(X_{\mathcal{F}}, B_n) := \{a \in \mathbb{A}^{B_n} \ : \ a \text{ does not see a pattern from } \mathcal{F}\}.$$

Friedland(1997) had proved that

$$\inf_n \frac{\log(|Loc(X, B_n)|)}{|B_n|} = h_{\text{top}}(X).$$

This helps us define an algorithm to approximate the entropy from above for all SFTs.

Periodic points: More details tomorrow

Let the set of periodic points of period $2n$ be give by

$$Per(X,n) := \{x \in X \ : \ \sigma^{2n\vec{e}_i}(x) = x \text{ for all } 1 \leq i \leq d\}.$$

Friedland(1997) also proved that for nearest neighbour SFTs that there exists $c > 0$ such that

$$\sup_n \frac{\log(|Per(X,n)|) - c|B_n \setminus B_{n-1}|}{|B_n|} \leq h_{\text{top}}(X).$$

To be able to approximate the entropy from below it is enough to know that

$$\lim_n \frac{\log(|Per(X,n)|)}{|B_n|} = h_{\text{top}}(X).$$

Periodic points are awesome

Theorem

If for an SFT X,

$$\lim_n \frac{\log(|Per(X,n)|)}{|B_n|} = h_{top}(X).$$

Then $h_{top}(X)$ can be approximated from above and below by an algorithm. In other words it is computable.

Theorem

For hom-shifts X_H^d, there exists $c' > 0$ such that

$$\frac{|Per(X_H^d, n)|}{|\mathcal{L}(X_H^d, B_n)|} \geq e^{-c' n^{d-1}}.$$

We will discuss all of this and give a proof in the next talk.

What do we need to prove universality: The main combinatorial estimate

$$Checker(X_H^d, n) := \{a : B_n \to H \ : \ a|_{B_n \setminus B_{n-1}} \text{ uses only two symbols}\}.$$

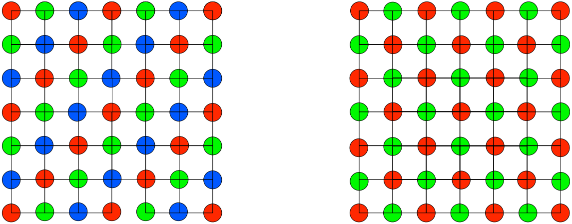

Figure : The left is not an element of Checker while the Right one is an element of Checker

What do we need to prove universality: The main combinatorial estimate

The main combinatorial component to prove universality is the following:

Theorem (Chandgotia and Meyerovitch 2019)

There exists $c'' > 0$ *such that*

$$\frac{|Checker(X_H^d, n)|}{|\mathcal{L}(X_H^d, B_n)|} \geq e^{-cn^{d-1}}.$$

Recall for the computability of the entropy we had to prove

$$\frac{|Per(X_H^d, n)|}{|\mathcal{L}(X_H^d, B_n)|} \geq e^{-c'n^{d-1}}.$$

More on this next time!